Exact Spherical Wave Solutions
to
Maxwell's Equations
with Applications

Guy G. Silvestri

A dissertation submitted to the faculty of the
Oregon Graduate Institute of Science & Technology
in partial fulfillment of the
requirements for the degree
Doctor of Philosophy
in
Applied Physics

January 1993

The dissertation "Exact Spherical Wave Solutions to Maxwell's Equations with Applications" by Guy G. Silvestri has been examined and approved by the following Examination Committee:

Raj Solanki, Dissertation Advisor
Associate Professor

V. S. Rao Gudimetla
Assistant Professor

Anthony E. Bell
Associate Professor

Norman Franzen
Tektronix, Incorporated

Table of Contents

Table of Contents (Cont.)

Table of Contents (Cont.)

List of Figures

ABSTRACT

Exact Spherical Wave Solutions to Maxwell's Equations with Applications

Guy G. Silvestri
Dissertation Advisor: Raj Solanki

Electromagnetic radiation from bounded sources represent an important class of physical problems that can be solved for exactly. However, available texts on this subject almost always resort to approximate solution techniques that not only obscure the essential features of the problem but also restrict application to limited ranges of observation.

This dissertation presents exact solutions for this important class of problems and demonstrates how these solutions can be applied to situations of genuine physical interest, in particular, the design of device structures with pre-specified emission characteristics.

The strategy employed is to solve Maxwell's equations in the spherical coordinate system. In this system, fundamental parameters such as electric and magnetic multipole moments fall out quite naturally. Expressions for radiated power, force, and torque assume especially illuminating and simple forms when expressed in terms of these multipole moments. All solutions are derived *ab initio* using first-principles arguments exclusively. Two operator-equations that receive particularly detailed treatment are the vector Helmholtz equation for the time-independent potential $\vec{a}$ and the "covariant divergence" equation for

the energy-momentum-stress tensor $T^{\mu\nu}$.

An application of classical formulas, as modified by the requirements of statistical mechanics, to the case of heated blackbodies leads to inquiries into the foundations of quantum mechanics and their relation to classical field theory.

An application of formulas to various emission structures (spherically-shaped antennas, surface diffraction gratings, collimated beams) provides a basis upon which to characterize these structures in an exact sense, and, ultimately, to elicit clues as to their optimum design.

CHAPTER I
INTRODUCTION

Maxwell's equations for the electromagnetic field are of paramount importance in physics. This thesis explores exact solutions to these equations in charge-free regions of space surrounding a bounded source. Explicit expressions for various conserved quantities assume surprisingly simple forms when expressed in terms of these exact solutions. These explicit expressions provide insights into the character of the Maxwellian field, and some practical consequences are qualitatively explored. Some historical background might prove informative.

Monochromatic electromagnetic fields of angular frequency ω exterior to a radiating body must satisfy appropriate boundary conditions at infinity. Specifically, the fields must assume the form of an outwardly-expanding spherical wave of radius ct, where c is the speed of light in vacuum and t is the time elapsed. Furthermore, to assure that the evolution of energy and momentum at large distances from the source remain within physically allowable limits, the fields are required to be asymptotically proportional to $1/r$. Any other radial dependence will lead to diverging energy expressions at infinity or, what is equally undesirable, vanishing energy expressions at infinity. Both the above requirements are subsumed in the mathematical statement that the radiating fields must approach $e^{i(kr-\omega t)}/r$ as $r \to \infty$, where k, the wave number, equals ω/c. The familiar plane wave and cylindrical wave solutions, so familiar from cavity mode theory, are thus disallowed in this formulation. However, when Maxwell's equations are solved in spherical coordinates, electrodynamic fields with the desired asymptotic characteristics are automatically obtained. In addition, many features of Maxwell's electrodynamics that exist in a somewhat camoflouged state when expressed in the Cartesian system become pronouncedly obvious when expressed in the spherical system. Thus, the merits of using this system surpass one's initial expectations from it. A detailed discussion of this system, along with its generalization to four dimensions, provides the impetus for Chapters IV and V of this investigation.

The primary task at hand is to obtain exact (spherical) solutions for the electric and magnetic fields in charge-free space. As is demonstrated in virtually every E & M textbook, this task is mathematically equivalent to solving the scalar and vector Helmholtz equations for the potentials Φ and $\vec{A}$, respectively. This equivalence, along with a full-blown exposition of a solution technique for the vector Helmholtz equation, comprises Chapters II and III of this dissertaion. The course of action outlined in Chapter III is by no means unique; investigators have occupied themselves with this problem for decades and various solution techniques have been devised [1–8]. A review of these approaches will provide some historical context.

The spherical solutions for $\vec{E}$ and $\vec{B}$ appear to have been completely worked out in the 1950's by several authors working independently. Two major schools of thought seem to have gained pre-eminence, namely, the method of vector spherical harmonics as championed by Blatt and Weisskopf[1], Hill[2], and the Russian authors[3,4], versus the method of Debye potentials as initially given by Bouwkamp and Casimir[5] and developed further by Nisbet[6].

One cautionary note should be made about these earlier papers. The calculational techniques are not for the faint of heart, and typically become extraordinarily difficult to follow. The Russian authors in particular presume a thorough knowledge of the techniques of group theory and creation/annihilation operators, among other things. The path-breaking paper by Bouwkamp and Casimir presumes facility with Green's functions. The derivations can oftentimes become so abstract that it is not certain until only the last moment that something concrete is going to fall out from them.

Nevertheless, the final $\vec{E}$ and $\vec{B}$ expressions, regardless of solution technique, found their way into the textbooks of the time, notably, Stratton[7] and Panofsky and Phillips[8]. The presentation by Panofsky and Phillips is particularly recommended for its clear and readable exposition of the Debye potential method. These authors were most likely already familiar with the exact solutions and were thus in a position to devise a more succinct method of deriving them. (Elegant derivations are usually possible when one knows beforehand what the solution is going to be.)

The ultimate solutions for monochromatic $\vec{E}$ and $\vec{B}$ are given in upcoming equations (III.F.11 thru 16) of this dissertation. These solutions are derived *ab initio* in Chapter III for the benefit of those who have not seen or have not been able to follow their derivations elsewhere. The derivation utilized in this report relies solely on knowledge of standard solution techniques for partial differential equations and mathematical identities involving the spherical Hankel, associated Legendre, and trigonometric functions. These special-function identities are provided in tabular form in the Appendix for easy reference.

The next major section of this report comprises Chapters IV and V, where the fundamental objects of mathematical physics, *viz.*, scalars, vectors, and tensors, are given their representation in the spherical coordinate system. Manipulation of tensor quantities in the spherical system is not always as simple-minded as it is in the Cartesian system, primarily because the spherical unit vectors $(\hat{r}, \hat{\theta}, \hat{\phi})$ are themselves functions of position, and partial derivatives of expressions containing these vectors must account for this fact. This point is driven home repeatedly in these two math-intensive chapters. Consequences of this state of affairs are explored extensively.

Differential identities such as divergence, gradient, and curl assume somewhat complicated forms in the spherical system, and it is important to know how to generate and handle them. In the language of tensor analysis, it is possible to express the seemingly complicated formulas of the spherical system (or any curvilinear system, for that matter) in an especially coherent manner. As such, this formalism is utilized throughout. To make sure that all formulas are fully understood, matrix quantities are written out in their entirety, even though doing so makes many formulas unwieldy-looking. But my experience has been that over-utilization of tensor shorthands, although providing for great notational compactness, obscures many essential truths. Therefore, matrix expressions have been written in full whenever necessary.

Other mathematical operations have not been treated so meticulously. The majority of partial differentiations and matrix multiplications are *not* typed out explicitly because it is felt that most readers can perform these operations for themselves. There is no need to boggle the developments with long-winded arithmetic if the essential results can be obtained without it.

There has been a conscientious attempt to adhere as closely as possible to the notation and sign conventions of Jackson[9], although this puts several formulas at odds with authors such as Arfken[10] and others. The approach that has been (independently) forwarded in this investigation strongly parallels that of Becker[11], who covers the topic of tensors in Part A of his comprehensive text on electromagnetics. Other good references are the previously mentioned Arfken[10] and the classic by Lovelock and Rund[12].

The subject of tensor analysis is broad and every author seems to have his own way of doing it. Further, the subject seems to have undergone some evolution over the years. After committing all the expressions of this dissertaion to print, it was discovered that "old" definitions for vector and tensor had been utilized, so definitions (IV.A.3) and (IV.A.6) disagree with modern authors. The major point of departure is that the scale factors $1/r$ and $1/r\sin\theta$ have been incorporated directly onto the θ- and ϕ-components of the *vector* (or *tensor*) in this investigation, whereas modern authors would leave these factors inside the transformation *matrix*. (Refer to equations (IV.A.3) and (IV.A.6) for details: Contrast these equations with those of any modern author. The determinant of the transformation matrix used in this report is 1; the determinant of other authors' transformation matrix would be $1/r^2\sin\theta$.) Rather than re-casting my old-style formulas into modern format, it was decided to let them stand as originally written because they make the transformation between Cartesian and spherical tensors simpler to comprehend and work with. The "modern" definitions for transformation matrix are directed with an eye towards applications in non-Euclidean spaces, a subject that is not ventured into in this investigation.

Once the mathematical preliminaries have been taken care of, attention is re-focussed back onto the pertinent physics. The mathematical framework established in Chapters IV and V is used to derive conservation laws for the electromagnetic system. The conserved quantities examined in this investigation are those that can be expressed quadratically in the field variables. (There are also conserved quantities of non-quadratic order, but these seem not to possess physical significance.) The Maxwell energy-momentum-stress tensor is used as the vehicle from which to launch the discussion. The mathematical formulation of this tensor is included in most E & M texts[4,9,11]. It therefore merely suffices to quote the tensor directly rather than devote unnecessary text to it. The interested reader can refer to the aforementioned texts for further insights into this topic.

Conservation laws in physics are of fundamental importance not only because they single out parameters that are of genuine physical interest (energy and momentum, for instance, rather than phlogiston or ether wind), but also because they guide one to "correct" formulations physical law (Newton's laws of motion versus Ptolemy's rotating epicycles, for instance). As such, conservation laws receive a great deal of emphasis in contemporary expositions of physical theory, and electrodynamics is no exception.

The definitions of electromagnetic energy, momentum, and angular momentum seem to have been firmly established by the turn of this century, and appropriate conservation laws for each of the above seven quantities were in general usage by then[11]. (*Seven* quantities because momentum and angular momentum are vectors.) The existence of additional conserved quantities has been a concern of more recent authors, all using a variety of techniques to discover them.

The presently favored methods of deriving conservation laws are not always intuitive, nor easy. Perhaps the most frequently-used approach[12] is to pre-assume some sort of invariance property for the physical laws being investigated (translation, rotation, or reflection invariance being typical), and then make use of Noether's Theorem to derive conservation laws that correspond to each invariance property (momentum, angular momentum, and parity being the conserved quantities corresponding to the three forms of invariance listed above). The difficulty with this approach is that *all* allowable invariance properties for the laws under investigation are usually not easy to ascertain. The simplest invariances are usually easy to spot, which explains why conservation laws for energy, momentum, and angular momentum typically far precede conservation laws for other parameters. More general, albeit more abstract, approaches are often necessary to extract additional conservation laws.

The definitive work by Fulton[13] and Rohrlich[14] established that Maxwell's equations in charge-free space display *conformal* invariance, that is to say that *angles* between vectors are preserved under given transformations, and that fifteen conserved quantities should be expected as a result. However, explicit expressions for all fifteen quantities have not been given. A simpler derivation technique outlined in Chapter V of this report provides independent verification that fifteen conserved quantities are indeed obtained from the electromagnetic energy-momentum-stress tensor. An added bonus is that explicit expressions for each of the fifteen conserved quantities are obtained in a deductive, non-arbitrary manner. No recourse whatsoever is made to the tools of Group Theory or Conformal Invariants. No variational techniques are invoked, thus obviating the need for Noether's Theorem. All that is required is familiarity with the partial differential behavior of the spherical unit vectors $(\hat{r}, \hat{\theta}, \hat{\phi})$ as given in Chapter V and the ability to handle partial differential equations. The fifteen conserved quantities implied by the electromagnetic energy-momentum-stress tensor fall out straightforwardly, with no initial guesswork required.

Chapter VI represents a synthesis of all previous chapters. The field solutions of Chapter III are incorporated into the conservation laws of Chapter V to obtain expressions for the radiated fluxes of energy, momentum, and angular momentum in terms of the electric and magnetic multipole moments of the given source. It should be stressed that these expressions are all *exact*, and fall out as inevitable consequences of Maxwell's equations. If these derived energy or momentum expressions are for some reason deemed unsuitable (due to some as-yet undefined aspect that renders them incorrect from a theoretical or practical viewpoint), it means that Maxwell's equations themselves require modification, and not the attendant conservation laws. Since Maxwell's equations have so valiantly withstood the test of time, it seems unlikely that they, or the derived energy and momentum expressions of this investigation, will require any modification at all. These formulas are versatile enough to even describe some simple quantum phenomena, as in fact is done in Chapter VII.

One of the more interesting results of the derived formulas of Chapter VI is the relationship between the total flux of electromagnetic energy and total flux of z-component of angular momentum. The relationship is highly reminiscent of the corresponding relation postulated in quantum mechanics. This relationship between energy and z-component of angular momentum has not gone unnoticed by other authors. Blatt and Weisskopf[1], Rohrlich[14], Heitler[15], and Panofsky and Phillips[8] all make mention of the relationship to the degree that their individual formalisms allow. In the exact formulation of Chapter VI, the relations is demonstrated conclusively. No approximations have been made to derive any of the formulas of this chapter; thus, the relation is shown to hold quite rigorously.

This relation, in fact, is the germ that spawns the discussion of Chapter VII, where Maxwellian expressions for radiated energy and angular momentum are used in tandem with the formulas of classical statistical mechanics to explain the emission characterics of radiating blackbodies. The fact that classical formulas are even remotely applicable to this most prototypical of quantum systems is something of a hint at the profundity of the Maxwellian expressions. These expressions seemingly contain within them seeds for a comprehensive review of quantized theories of the electromagnetic field.

The discussion of Chapter VII is very much in the spirit of Max Planck who, although being the grandfather of quantum mechanics, never quite resigned himself to its latter-day developments. Planck's semi-classical approach to the blackbody problem, as supplemented by the electromagnetic formulas of Chapter VI, is utilized throughout. His persistent belief that the energy quanta $\hbar\omega$ follows from classical considerations may yet be realized, even if only partially. (This theoretical pursuit is not the purpose of this investigation; the topic will be taken up elsewhere.)

The concluding Chapter VIII serves to demonstrate practical applications of the formulas of Chapter VI. Three examples are used. The first example involves a spherically-shaped antenna, a non-practical situation to be sure, but one that lends itself readily to solution in the spherical coordinate system. This worked example is included mainly for pedagogical purposes, it being a simple-to-follow demonstration of how the energy, momentum, and angular momentum formulas of the previous chapters are solved for and utilized.

The second example is a rough-hewn attempt at determining the shape of a charge structure that radiates near-monochromatically (*i.e.*, a surface diffraction grating). The exact structure is not actually obtained because computer calculations and additional refinements to the theory are going to be required. But the preliminary mathematics are established. The material presented in this section should be considered as first-cut only.

The third example is a qualitative discussion about modelling collimated beams spherically. The typical approach is to simply model the beam using plane waves. But recall that plane waves do not attenuate at infinity, and thus are disallowed as solutions for radiation modes. Although it must be conceded that plane waves are exceptionally well-suited as *approximate* solutions for electromagnetic waves close to the radiating source ("near field" solutions), extrapolating these plane wave solutions to distances far from the source is tantamount to inviting infinite energy and momentum fluxes, a clearly undesirable situation. If these infinite-energy solutions are to be avoided, a spherical solution that closely approximates plane waves must be utilized instead. This third and last section of Chapter VIII addresses this problem.

CHAPTER II

REVIEW OF MAXWELL'S EQUATIONS

Radiative electromagnetic fields are best expressed in terms of spherical coordinates (r,θ,ϕ) because boundary conditions at $r \to \infty$ are automatically fulfilled for solutions expressed in this system.

The intent of these next few chapters is to demonstrate a solution technique for Maxwell's equations in the homogeneous, monochromatic case. The solutions obtained using spherical coordinates provide insights that are likely to overlooked when using other coordinate systems.

By way of review, we have Maxwell's equations:

(II.1)

$$\nabla \cdot \vec{E} = 4\pi\rho \quad \text{(a)}$$

$$\nabla \cdot \vec{B} = 0 \quad \text{(b)}$$

$$\nabla \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 \quad \text{(c)}$$

$$\nabla \times \vec{B} - \frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{4\pi}{c}\vec{J} \quad \text{(d)}$$

and

$$\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0 \quad \text{(II.2)}$$

It is standard practice to decouple Maxwell's Equations by defining scalar and vector potentials $(\Phi, \vec{A})$ for which we obtain:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -4\pi\rho \tag{II.3}$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\frac{4\pi}{c}\vec{J} \tag{II.4}$$

and

$$\nabla\cdot\vec{A} = -\frac{1}{c}\frac{\partial\Phi}{\partial t} \tag{II.5}$$

The field vectors $\vec{E}$ and $\vec{B}$ are recovered from the $(\Phi, \vec{A})$ potentials via the relations:

$$\vec{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \tag{II.6}$$

$$\vec{B} = \nabla\times\vec{A} \tag{II.7}$$

If solutions are to be restricted to the monochromatic case, $e^{\pm i\omega t}$ time dependencies are pre-imposed upon the various scalar and vector quantities as follows:

$$\vec{E}(\vec{x}, t) = \frac{1}{2}\left(\vec{e}\,e^{-i\omega t} + \vec{e}^{*}e^{i\omega t}\right) \tag{II.8}$$

$$\vec{B}(\vec{x}, t) = \frac{1}{2}\left(\vec{b}\,e^{-i\omega t} + \vec{b}^{*}e^{i\omega t}\right) \tag{II.9}$$

$$\rho(\vec{x}, t) = \frac{1}{2}\left(q e^{-i\omega t} + q^{*}e^{i\omega t}\right) \tag{II.10}$$

$$\vec{J}(\vec{x}, t) = \frac{1}{2}\left(\vec{j}\,e^{-i\omega t} + \vec{j}^{*}e^{i\omega t}\right) \tag{II.11}$$

The time-independent, complex-valued components $\vec{e}$, $\vec{b}$, q, and $\vec{j}$ satisfy their own set of time independent Maxwell equations:

$$\nabla\cdot\vec{e} = 4\pi q \quad \text{(a)}$$

$$\nabla\cdot\vec{b} = 0 \quad \text{(b)}$$

$$\nabla\times\vec{e} - ik\vec{b} = 0 \quad \text{(c)}$$

$$\nabla\times\vec{b} + ik\vec{e} = \frac{4\pi}{c}\vec{j} \quad \text{(d)}$$

(II.12)

and

$$\nabla\cdot\vec{j} - i\omega q = 0 \qquad \text{(II.13)}$$

$$\text{where} \quad k = \frac{\omega}{c}$$

Decoupling is achieved as before by resorting to scalar and vector potentials:

$$\Phi(\vec{x}\,,\,t) = \frac{1}{2}\left(\psi e^{-i\omega t} + \psi^* e^{i\omega t}\right) \qquad \text{(II.14)}$$

$$\vec{A}(\vec{x}\,,\,t) = \frac{1}{2}\left(\vec{a}\, e^{-i\omega t} + \vec{a}^* e^{i\omega t}\right) \qquad \text{(II.15)}$$

for which:

$$\nabla^2\psi + k^2\psi = -4\pi q \qquad \text{(II.16)}$$

$$\nabla^2\vec{a} + k^2\vec{a} = -\frac{4\pi}{c}\vec{j} \qquad \text{(II.17)}$$

and

$$\nabla\cdot\vec{a} = ik\psi \qquad \text{(II.18)}$$

Also:

$$\vec{e} = -\nabla\psi + ik\vec{a} \qquad (II.19)$$

$$\vec{b} = \nabla\times\vec{a} \qquad (II.20)$$

In source-free regions of space, $(\rho, \vec{J}) = 0$, and one obtains:

$$\nabla^2\psi + k^2\psi = 0 \qquad (II.21)$$

$$\nabla^2\vec{a} + k^2\vec{a} = 0 \qquad (II.22)$$

and

$$\nabla\cdot\vec{a} = ik\psi \qquad (II.23)$$

$\psi(\vec{x})$ therefore satisfies the scalar Helmholtz equation and $\vec{a}(\vec{x})$ satisfies the vector Helmholtz equation. Expressed in spherical coordinates, the $\nabla^2\psi + k^2\psi = 0$ equation becomes:

$$\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + k^2\psi = 0 \qquad (II.24)$$

Standard separation-of-variables techniques lead to the well-documented solution:

$$\psi(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_{lm}\, h_l^{(1)}(kr)\, P_l^m(\cos\theta)\, e^{im\phi} \qquad (II.25)$$

where c_{lm} = expansion coefficient

$h_l^{(1)}(kr)$ = Spherical Hankel Function of 1st Kind

$P_l^m(\cos\theta)$ = Associated Legendre Function of 1st Kind

Also, see note below.†

Similarly, $\vec{a}$ satisfies the vector Helmholtz equation, $\nabla^2\vec{a} + k^2\vec{a} = 0$, which expressed in spherical coordinates becomes:

$$\hat{r}\left(\nabla^2 a_r + k^2 a_r - \frac{2}{r^2}a_r - \frac{2}{r^2}\frac{\partial a_\theta}{\partial\theta} - \frac{2\cos\theta}{r^2\sin\theta}a_\theta - \frac{2}{r^2\sin\theta}\frac{\partial a_\phi}{\partial\phi}\right)$$
$$+ \hat{\theta}\left(\nabla^2 a_\theta + k^2 a_\theta - \frac{1}{r^2\sin^2\theta}a_\theta + \frac{2}{r^2}\frac{\partial a_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial a_\phi}{\partial\phi}\right)$$
$$+ \hat{\phi}\left(\nabla^2 a_\phi + k^2 a_\phi - \frac{1}{r^2\sin^2\theta}a_\phi + \frac{2}{r^2\sin\theta}\frac{\partial a_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial a_\theta}{\partial\phi}\right) = 0 \quad \text{(II.26)}$$

The forbidding appearance of this vector equation possibly explains why exact solutions are so infrequently found in the literature.

A general solution of this equation will be derived in Chapter III.

Before embarking on solution techniques, it is essential that several mathematical identities be put at our immediate disposal. Hence, a compendium of relevant formulas is provided in the Appendix.

---

† There are also "2nd Kind" spherical Hankel $h_l^{(2)}(kr)$ and associated Legendre $Q_l^m(\cos\theta)$ solutions to the $\nabla^2\psi + k^2\psi = 0$ equation, but the $h_l^{(1)}(kr)P_l^m(\cos\theta)$ solutions have been purposelly singled out because:

* ψ must behave as an *outgoing* spherical wave at $r \to \infty$. This eliminates the $h_l^{(2)}(kr)$ solution.
* ψ must be finite valued at $\cos\theta = 1$. This eliminates the $Q_l^m(\cos\theta)$ solution.

CHAPTER III

SOLUTION OF THE VECTOR HELMHOLTZ EQUATION IN SPHERICAL COORDINATES

A.) Initial Statement of Problem and Equivalent Formulation

Recall from Chapter II the vector equation of interest:

$$\nabla^2 \vec{a} + k^2 \vec{a} = 0 \tag{III.A.1}$$

subject to

$$\nabla \cdot \vec{a} = ik\psi \tag{III.A.2}$$

where

$$\psi = c_{lm} h_l^{(1)}(kr)\, P_l^m(\cos\theta)\, e^{im\phi} \tag{III.A.3}$$

The vector operation $\nabla^2 \vec{a}$ is defined from the identity:

$$\nabla^2 \vec{a} = \nabla(\nabla \cdot \vec{a}) - \nabla \times \nabla \times \vec{a} \tag{III.A.4}$$

where from (App.A.11) we have:

$$\nabla^2 \vec{a} = \frac{\vec{r}}{r^2} \nabla^2 (\vec{r} \cdot \vec{a}) - \frac{2\vec{r}}{r^2} \nabla \cdot \vec{a} + \frac{2\vec{r}}{r^2} \times (\nabla \times \vec{a}) - \frac{\vec{r}}{r^2} \times \nabla^2 (\vec{r} \times \vec{a}) \tag{III.A.5}$$

Take the dot product and cross product of the above relation with $\vec{r}$ to obtain:

$$\vec{r}\cdot\nabla^2\vec{a} \;=\; \nabla^2(\vec{r}\cdot\vec{a}) \;-\; 2\nabla\cdot\vec{a} \qquad \text{(III.A.6)}$$

$$\vec{r}\times\nabla^2\vec{a} \;=\; \nabla^2(\vec{r}\times\vec{a}) \;-\; 2\nabla\times\vec{a} \qquad \text{(III.A.7)}$$

Since it is required that:

$$\nabla^2\vec{a} \;=\; -k^2\vec{a}$$

$$\nabla\cdot\vec{a} \;=\; ik\psi$$

one obtains the equivalent formulation of the vector Helmholtz problem:

$$(\nabla^2 + k^2)(\vec{r}\cdot\vec{a}) = 2ik\psi \qquad \text{(III.A.8)}$$

$$(\nabla^2 + k^2)(\vec{r}\times\vec{a}) = 2\nabla\times\vec{a} \qquad \text{(III.A.9)}$$

B.) Two "particular" solutions for a_r

To determine the component a_r, utilize the first relation (III.A.8):

$$
\begin{aligned}
2ikc_{lm}\,h_l^{(1)}(kr)\;P_l^m(\cos\theta)\;e^{im\phi} &= (\nabla^2 + k^2)(\vec{r}\cdot\vec{a}\,) \\
&= (\nabla^2 + k^2)(ra_r) \qquad \text{(III.B.1)} \\
&= r(\nabla^2 + k^2)a_r + 2\frac{\partial a_r}{\partial r} + \frac{2}{r}a_r
\end{aligned}
$$

Consider a_r to be of the form:

$$
a_r = a_{l'm'}\,h_{l'}^{(1)}(kr)\;P_{p'}^{q'}(\cos\theta)\;e^{im'\phi} \qquad \text{(III.B.2)}
$$

Use equations (App.B2.1) and (App.C2.1) on this pre-assumed form for a_r to obtain:

$$
\begin{aligned}
(\nabla^2 + k^2)a_r &= a_{l'm'}\,r\left(\frac{l'(l'+1) - p'(p'+1)}{r^2} + \right. \qquad \text{(III.B.3)} \\
&\left. + \frac{q'^2 - m'^2}{r^2\sin^2\theta}\right)h_{l'}^{(1)}(kr)\;P_{p'}^{q'}(\cos\theta)\;e^{im'\phi}
\end{aligned}
$$

Plugging (III.B.3) into (III.B.1) leads to:

$$2ikc_{lm}\, h_l^{(1)}(kr)\; P_l^m(\cos\theta)\; e^{im\phi} = \tag{III.B.4}$$

$$= a_{l'm'}\left(\frac{l'(l'+1) - p'(p'+1)}{r} + \frac{q'^2 - m'^2}{r\sin^2\theta}\right) h_{l'}^{(1)}(kr)\; P_{p'}^{q'}(\cos\theta)\; e^{im'\phi}$$

$$+ \; 2a_{l'm'}\left(\frac{dh_{l'}^{(1)}(kr)}{dr} + \frac{h_{l'}^{(1)}(kr)}{r}\right) P_{p'}^{q'}(\cos\theta)\; e^{im'\phi}$$

Since the above relation must hold identically, we require strict equality of indices among all P_l^m and $e^{im\phi}$ terms. Hence:

$$q' = m$$

$$p' = l$$

$$m' = m$$

The P_l^m and $e^{im\phi}$ terms therefore cancel and one is left with:

$$2ikc_{lm}\, h_l^{(1)}(kr) = a_{l'm}\left(\frac{l'(l'+1) - l(l+1)}{r}\right) h_{l'}^{(1)}(kr) + \tag{III.B.5}$$

$$+ \; 2a_{l'm}\left(\frac{dh_{l'}^{(1)}(kr)}{dr} + \frac{h_{l'}^{(1)}(kr)}{r}\right)$$

Utilize equations (App.B2.8) or (App.B2.7) to replace the terms enclosed in the second set of parantheses:

$$2ikc_{lm}h_l^{(1)}(kr) = \begin{cases} a_{l'm}\left(\dfrac{l'(l'+1) - l(l+1)}{r}\right)h_{l'}^{(1)}(kr) + \\ \qquad + 2a_{l'm}\left((l'+1)\dfrac{h_{l'}^{(1)}(kr)}{r} - kh_{l'+1}^{(1)}(kr)\right) \\ \text{or:} \\ a_{l'm}\left(\dfrac{l'(l'+1) - l(l+1)}{r}\right)h_{l'}^{(1)}(kr) + \\ \qquad + 2a_{l'm}\left(kh_{l'-1}^{(1)}(kr) - l'\dfrac{h_{l'}^{(1)}(kr)}{r}\right) \end{cases} \tag{III.B.6}$$

To force equality in the first case, set:

$$l' = l-1$$

To force equality in the second case, set:

$$l' = l+1$$

These choices for l' in turn force:

$$a_{l'm} = \begin{cases} -ic_{lm} & \text{(First Case)} & \text{(III.B.7)} \\ ic_{lm} & \text{(Second Case)} & \text{(III.B.8)} \end{cases}$$

Thus, two independent "particular" solutions to the inhomogeneous equation $(\nabla^2 + k^2)(ra_r) = 2ikc_{lm}h_l^{(1)}(kr)\ P_l^m(\cos\theta)\ e^{im\phi}$ have been derived, namely:

$$a_r = \begin{cases} -ic_{lm}h_{l-1}^{(1)}(kr)\ P_l^m(\cos\theta)\ e^{im\phi} & \text{(III.B.9)} \\ ic_{lm}h_{l+1}^{(1)}(kr)\ P_l^m(\cos\theta)\ e^{im\phi} & \text{(III.B.10)} \end{cases}$$

C.) Two "particular" solutions for a^+ and a^-

To determine the components a_θ and a_ϕ, utilize the second relation (III.A.9):

$$\begin{aligned} 2\nabla\times\vec{a} &= (\nabla^2 + k^2)(\vec{r}\times\vec{a}) \\ &= (\nabla^2 + k^2)(r\hat{r}\times(a_r\hat{r} + a_\theta\hat{\theta} + a_\phi\hat{\phi})) \\ &= (\nabla^2 + k^2)(-ra_\phi\hat{\theta} + ra_\theta\hat{\phi}) \end{aligned}$$

Use equation (App.A.2) on the L.H.S. and (App.A.8) on the R.H.S. to obtain:

$$\begin{aligned} &\hat{r}\left(\frac{2}{r}\frac{\partial a_\phi}{\partial\theta} + \frac{2\cos\theta}{r\sin\theta}a_\phi - \frac{2}{r\sin\theta}\frac{\partial a_\theta}{\partial\phi}\right) \\ &+ \hat{\theta}\left(\frac{2}{r\sin\theta}\frac{\partial a_r}{\partial\phi} - 2\frac{\partial a_\phi}{\partial r} - \frac{2}{r}a_\phi\right) \\ &+ \hat{\phi}\left(2\frac{\partial a_\theta}{\partial r} + \frac{2}{r}a_\theta - \frac{2}{r}\frac{\partial a_r}{\partial\theta}\right) = \end{aligned}$$

$$\begin{aligned} &= \hat{r}\left(\frac{2}{r}\frac{\partial a_\phi}{\partial\theta} + \frac{2\cos\theta}{r\sin\theta}a_\phi - \frac{2}{r\sin\theta}\frac{\partial a_\theta}{\partial\phi}\right) \\ &- \hat{\theta}\left(\left(\nabla^2 - \frac{1}{r^2\sin^2\theta} + k^2\right)(ra_\phi) + \frac{2\cos\theta}{r\sin^2\theta}\frac{\partial a_\theta}{\partial\phi}\right) \\ &+ \hat{\phi}\left(\left(\nabla^2 - \frac{1}{r^2\sin^2\theta} + k^2\right)(ra_\theta) - \frac{2\cos\theta}{r\sin^2\theta}\frac{\partial a_\phi}{\partial\phi}\right) \end{aligned}$$

Equate coefficients of $\hat{\theta}$ and $\hat{\phi}$, divide through by r, and make use of relation (App.A.7) to obtain:

$$\left(\nabla^2 - \frac{1}{r^2\sin^2\theta} + k^2\right)a_\phi + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial a_\theta}{\partial \phi} = -\frac{2}{r^2\sin\theta}\frac{\partial a_r}{\partial \phi} \qquad \text{(III.C.1)}$$

$$\left(\nabla^2 - \frac{1}{r^2\sin^2\theta} + k^2\right)a_\theta - \frac{2\ \cos\theta}{r^2\sin^2\theta}\frac{\partial a_\phi}{\partial \phi} = -\frac{2}{r^2}\frac{\partial a_r}{\partial \theta} \qquad \text{(III.C.2)}$$

Multiply the first equation by i and add and subtract to the second equation to obtain:

$$\left(\nabla^2 - \frac{1}{r^2\sin^2\theta} + \frac{2i\cos\theta}{r^2\sin^2\theta}\frac{\partial}{\partial\phi} + k^2\right)a^+ = -\frac{2}{r^2}\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right)a_r \qquad \text{(III.C.3)}$$

$$\left(\nabla^2 - \frac{1}{r^2\sin^2\theta} - \frac{2i\cos\theta}{r^2\sin^2\theta}\frac{\partial}{\partial\phi} + k^2\right)a^- = -\frac{2}{r^2}\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right)a_r \qquad \text{(III.C.4)}$$

where
$$a^+ = a_\theta + ia_\phi \qquad \text{(III.C.5)}$$

$$a^- = a_\theta - ia_\phi \qquad \text{(III.C.6)}$$

Define tilde-functions independent of m:

$$a_r = \tilde{a}_r e^{im\phi} \qquad \text{(III.C.7)}$$

$$a_\theta = \tilde{a}_\theta e^{im\phi} \qquad \text{(III.C.8)}$$

$$a_\phi = \tilde{a}_\phi e^{im\phi} \qquad \text{(III.C.9)}$$

and:

$$\tilde{a}^+ = \tilde{a}_\theta + i\tilde{a}_\phi \qquad \text{(III.C.10)}$$

$$\tilde{a}^- = \tilde{a}_\theta - i\tilde{a}_\phi \qquad \text{(III.C.11)}$$

The pair of equations (III.C.3) and (III.C.4) become:

$$\left(\nabla^2 - \frac{m^2 + 2m\cos\theta + 1}{r^2\sin^2\theta} + k^2\right)\tilde{a}^+ = -\frac{2}{r^2}\left(\frac{\partial}{\partial\theta} - \frac{m}{\sin\theta}\right)\tilde{a}_r \qquad \text{(III.C.12)}$$

$$\left(\nabla^2 - \frac{m^2 - 2m\cos\theta + 1}{r^2\sin^2\theta} + k^2\right)\tilde{a}^- = -\frac{2}{r^2}\left(\frac{\partial}{\partial\theta} + \frac{m}{\sin\theta}\right)\tilde{a}_r \qquad \text{(III.C.13)}$$

From (III.B.9) and (III.B.10), two options for $\tilde{a}_r$ are available. We restrict attention exclusively to the second of these two options for the developments that follow:

$$\tilde{a}_r = i c_{lm} h_{l+1}^{(1)}(kr) P_l^m(\cos\theta) \qquad \text{(III.C.14)}$$

As with the a_r component, two independent "particular" solutions for $\tilde{a}^+$ and $\tilde{a}^-$ will be derived. The first of the two $\tilde{a}^+$ solutions is derived as follows:

Collect equations (III.C.12) and (III.C.14) and then operate on the R.H.S. using equation (App.C2.13):

$$\left(\nabla^2 + k^2 - \frac{(m+1)^2}{r^2\sin^2\theta} + \frac{2m(1-\cos\theta)}{r^2\sin^2\theta}\right)\tilde{a}^+ = \tag{III.C.15}$$

$$= -\frac{2}{r^2}\left(\frac{\partial}{\partial\theta} - \frac{m}{\sin\theta}\right)\left(ic_{lm}\, h_{l+1}^{(1)}(kr) P_l^m(\cos\theta)\right)$$

$$= -2ic_{lm}\frac{h_{l+1}^{(1)}(kr)}{r^2}\left(\frac{1-\cos\theta}{\sin^2\theta}\right)\frac{1}{2l+1}\, \cdot$$

$$\cdot \left[lP_{l+1}^{m+1}(\cos\theta) + (2l+1)P_l^{m+1}(\cos\theta) + (l+1)P_{l-1}^{m+1}(\cos\theta)\right]$$

Assume a solution of the form

$$\tilde{a}^+ = ic_{lm}h_{l+1}^{(1)}(kr)\left[a_1 P_{l+1}^{m+1}(\cos\theta) + a_2 P_l^{m+1}(\cos\theta)\right]$$

where the constants a_1 and a_2 are to be determined.

Plug this assumed form for $\tilde{a}^+$ into the L.H.S. of (III.C.15), and use equations (App.B2.1) and (App.C2.1) to calculate the effect of the ∇^2 operation:

$$\left(\nabla^2+k^2-\frac{(m+1)^2}{r^2\sin^2\theta}+\frac{2m(1-\cos\theta)}{r^2\sin^2\theta}\right)\tilde{a}^+ =$$

$$= \; ic_{lm}\,a_1\left[\frac{(l+1)(l+2)-(l+1)((l+2)}{r^2} \;+\; \frac{2m(1-\cos\theta)}{r^2\sin^2\theta}\right]h^{(1)}_{l+1}(kr)P^{m+1}_{l+1}(\cos\theta)$$

$$+ \; ic_{lm}\,a_2\left[\frac{(l+1)(l+2)-l(l+1)}{r^2} \;+\; \frac{2m(1-\cos\theta)}{r^2\sin^2\theta}\right]h^{(1)}_{l+1}(kr)P^{m+1}_{l}(\cos\theta)$$

$$= \; 2ic_{lm}\frac{h^{(1)}_{l+1}}{r^2}\left(\frac{1-\cos\theta}{\sin^2\theta}\right)\Big[a_1 mP^{m+1}_{l+1}(\cos\theta) \; +$$

$$+ \; a_2(l+m+1)P^{m+1}_{l}(\cos\theta) \; +$$

$$+ \; a_2(l+1)\cos\theta P^{m+1}_{l}(\cos\theta)\Big]$$

Use equation (App.C2.4) on the third term inside the brackets and pull out the (2l+1) term that gets introduced into the denominator:

$$= \; 2ic_{lm}\frac{h^{(1)}_{l+1}(kr)}{r^2}\left(\frac{1-\cos\theta}{\sin^2\theta}\right)\frac{1}{2l+1} \; \cdot$$

$$\cdot \; \Big[\big(a_1 m(2l+1) \;+\; a_2(l+1)(l-m)\big)P^{m+1}_{l+1}(\cos\theta) \; +$$

$$+ \; a_2(2l+1)(l+m+1)P^{m+1}_{l}(\cos\theta) \; +$$

$$+ \; a_2(l+1)(l+m+1)P^{m+1}_{l-1}(\cos\theta)\Big]$$

Equate the above expression to the R.H.S. of (III.C.15) to obtain coefficient matching conditions for a_1 and a_2:

$$a_1 m(2l+1) + a_2(l+1)(l-m) = -l$$

$$a_2(2l+1)(l+m+1) = -(2l+1)$$

$$-a_2(l+1)(l+m+1) = -(l+1)$$

Solutions to the above trio of equations work out to be:

$$a_2 = \frac{-1}{(l+m+1)} \qquad \text{(III.C.16)}$$

$$a_1 = \frac{-1}{(l+m+1)} \qquad \text{(III.C.17)}$$

From which one obtains:

$$\tilde{a}^+ = \frac{-ic_{lm}}{(l+m+1)} h^{(1)}_{l+1}(kr)\Big[P^{m+1}_{l+1}(\cos\theta) + P^{m+1}_{l}(\cos\theta)\Big] \qquad \text{(III.C.18)}$$

The second of the two $\tilde{a}^+$ solutions is derived in close analogy with the first.

Once again, collect equations (III.C.12) and (III.C.14), but this time operate on the R.H.S. using equation (App.C2.14):

$$\left(\nabla^2 + k^2 - \frac{(m-1)^2}{r^2\sin^2\theta} - \frac{2m(1+\cos\theta)}{r^2\sin^2\theta}\right)\tilde{a}^+ = \tag{III.C.19}$$

$$= -\frac{2}{r^2}\left(\frac{\partial}{\partial\theta} - \frac{m}{\sin\theta}\right)\left(ic_{lm}\, h^{(1)}_{l+1}(kr)\, P_l^m(\cos\theta)\right)$$

$$= -2ic_{lm}\frac{h^{(1)}_{l+1}(kr)}{r^2}\left(\frac{1+\cos\theta}{\sin^2\theta}\right)\frac{1}{2l+1} \cdot$$

$$\cdot \Big[l(l-m+1)(l-m+2)P^{m-1}_{l+1}(\cos\theta) -$$

$$- (2l+1)(l-m+1)(l+m)P_l^{m-1}(\cos\theta) +$$

$$+ (l+1)(l+m-1)(l+m)P^{m-1}_{l-1}(\cos\theta)\Big]$$

Assume a solution of the form

$$\tilde{a}^+ = ic_{lm} h^{(1)}_{l+1}(kr)\left[b_1 P^{m-1}_{l+1}(\cos\theta) + b_2 P_l^{m-1}(\cos\theta)\right]$$

where the constants b_1 and b_2 are to be determined.

Plug this assumed form for $\tilde{a}^+$ into the L.H.S. of (III.C.19), and use equations (App.B2.1) and (App.C2.1) to calculate the effect of the ∇^2 operation:

$$\left(\nabla^2+k^2-\frac{(m-1)^2}{r^2\sin^2\theta}-\frac{2m(1+\cos\theta)}{r^2\sin^2\theta}\right)\tilde{a}^+ =$$

$$= ic_{lm}\, b_1\left[\frac{(l+1)(l+2)-(l+1)((l+2)}{r^2} - \frac{2m(1+\cos\theta)}{r^2\sin^2\theta}\right]h^{(1)}_{l+1}(kr)P^{m-1}_{l+1}(\cos\theta)$$

$$+ ic_{lm}\, b_2\left[\frac{(l+1)(l+2)-l(l+1)}{r^2} - \frac{2m(1+\cos\theta)}{r^2\sin^2\theta}\right]h^{(1)}_{l+1}(kr)P^{m-1}_{l}(\cos\theta)$$

$$= -2ic_{lm}\frac{h^{(1)}_{l+1}}{r^2}\left(\frac{1+\cos\theta}{\sin^2\theta}\right)\Big[b_1 m P^{m-1}_{l+1}(\cos\theta) -$$

$$- b_2(l-m+1)P^{m-1}_{l}(\cos\theta) +$$

$$+ b_2(l+1)\cos\theta P^{m-1}_{l}(\cos\theta)\Big]$$

Use equation (App.C2.4) on the third term inside the brackets and pull out the (2l+1) term that gets introduced into the denominator:

$$= -2ic_{lm}\frac{h^{(1)}_{l+1}(kr)}{r^2}\left(\frac{1+\cos\theta}{\sin^2\theta}\right)\frac{1}{2l+1} \cdot$$

$$\cdot \Big[\big(b_1 m(2l+1) + b_2(l+1)(l-m+2)\big)P^{m-1}_{l+1}(\cos\theta) -$$

$$- b_2(2l+1)(l-m+1)P^{m-1}_{l}(\cos\theta) +$$

$$+ b_2(l+1)(l+m-1)P^{m-1}_{l-1}(\cos\theta)\Big]$$

Equate the above expression to the R.H.S. of (III.C.19) to obtain coefficient matching conditions for b_1 and b_2:

$$\begin{aligned} b_1 m(2l+1) + b_2(l+1)(l-m+2) &= l(l-m+1)(l-m+2) \\ b_2(2l+1)(l-m+1) &= (2l+1)(l-m+1)(l+m) \\ b_2(l+1)(l+m-1) &= (l+1)(l+m-1)(l+m) \end{aligned}$$

Solutions to the above trio of equations work out to be:

$$b_2 = (l+m) \qquad \text{(III.C.20)}$$

$$b_1 = -(l-m+2) \qquad \text{(III.C.21)}$$

From which one obtains:

$$\tilde{a}^+ = ic_{lm} h_{l+1}^{(1)}(kr)\Big[-(l-m+2)P_{l+1}^{m-1}(\cos\theta) + (l+m)P_l^{m+1}(\cos\theta)\Big] \qquad \text{(III.C.22)}$$

A completely parallel development for the $\tilde{a}^-$ component proceeds as follows:

Collect equations (III.C.13) and (III.C.14) and then operate on the R.H.S. using equation (App.C2.15):

$$\left(\nabla^2 + k^2 - \frac{(m+1)^2}{r^2\sin^2\theta} + \frac{2m(1+\cos\theta)}{r^2\sin^2\theta}\right)\tilde{a}^- = \qquad \text{(III.C.23)}$$

$$= -\frac{2}{r^2}\left(\frac{\partial}{\partial\theta} + \frac{m}{\sin\theta}\right)\left(ic_{lm}\, h_{l+1}^{(1)}(kr)\, P_l^m(\cos\theta)\right)$$

$$= 2ic_{lm}\frac{h_{l+1}^{(1)}(kr)}{r^2}\left(\frac{1+\cos\theta}{\sin^2\theta}\right)\frac{1}{2l+1} \cdot$$

$$\cdot \left[lP_{l+1}^{m+1}(\cos\theta) - (2l+1)P_l^{m+1}(\cos\theta) + (l+1)P_{l-1}^{m+1}(\cos\theta)\right]$$

Assume a solution of the form

$$\tilde{a}^- = ic_{lm}\, h_{l+1}^{(1)}(kr)\left[a_1 P_{l+1}^{m+1}(\cos\theta) + a_2 P_l^{m+1}(\cos\theta)\right]$$

where the constants a_1 and a_2 are to be determined.

Plug this assumed form for $\tilde{a}^-$ into the L.H.S. of (III.C.23), and use equations (App.B2.1) and (App.C2.1) to calculate the effect of the ∇^2 operation:

$$\left(\nabla^2+k^2-\frac{(m+1)^2}{r^2\sin^2\theta}+\frac{2m(1+\cos\theta)}{r^2\sin^2\theta}\right)\tilde{a}^- =$$

$$= \ ic_{lm}\,a_1\left[\frac{(l+1)(l+2)-(l+1)((l+2)}{r^2} + \frac{2m(1+\cos\theta)}{r^2\sin^2\theta}\right]h^{(1)}_{l+1}(kr)P^{m+1}_{l+1}(\cos\theta)$$

$$+ \ ic_{lm}\,a_2\left[\frac{(l+1)(l+2)-l(l+1)}{r^2} + \frac{2m(1+\cos\theta)}{r^2\sin^2\theta}\right]h^{(1)}_{l+1}(kr)P^{m+1}_{l}(\cos\theta)$$

$$= \ 2ic_{lm}\frac{h^{(1)}_{l+1}}{r^2}\left(\frac{1+\cos\theta}{\sin^2\theta}\right)\Big[a_1 mP^{m+1}_{l+1}(\cos\theta) +$$

$$+ \ a_2(l+m+1)P^{m+1}_{l}(\cos\theta) -$$

$$- \ a_2(l+1)\cos\theta P^{m+1}_{l}(\cos\theta)\Big]$$

Use equation (App.C2.4) on the third term inside the brackets and pull out the (2l+1) term that gets introduced into the denominator:

$$= \ 2ic_{lm}\frac{h^{(1)}_{l+1}(kr)}{r^2}\left(\frac{1+\cos\theta}{\sin^2\theta}\right)\frac{1}{2l+1} \ \cdot$$

$$\cdot \ \Big[\Big(a_1 m(2l+1) - a_2(l+1)(l-m)\Big)P^{m+1}_{l+1}(\cos\theta) +$$

$$+ \ a_2(2l+1)(l+m+1)P^{m+1}_{l}(\cos\theta) -$$

$$- \ a_2(l+1)(l+m+1)P^{m+1}_{l-1}(\cos\theta)\Big]$$

Equate the above expression to the R.H.S. of (III.C.23) to obtain coefficient matching conditions for a_1 and a_2:

$$a_1 m(2l+1) - a_2(l+1)(l-m) = l$$

$$a_2(2l+1)(l+m+1) = -(2l+1)$$

$$-a_2(l+1)(l+m+1) = (l+1)$$

Solutions to the above trio of equations work out to be:

$$a_2 = \frac{-1}{(l+m+1)} \qquad \text{(III.C.24)}$$

$$a_1 = \frac{1}{(l+m+1)} \qquad \text{(III.C.25)}$$

From which one obtains:

$$\tilde{a}^- = \frac{ic_{lm}}{(l+m+1)} h_{l+1}^{(1)}(kr)\left[P_{l+1}^{m+1}(\cos\theta) - P_l^{m+1}(\cos\theta)\right] \qquad \text{(III.C.26)}$$

Similarly, the second of the two $\tilde{a}^-$ solutions is derived as follows:

Collect equations (III.C.13) and (III.C.14), and operate on the R.H.S. using equation (App.C2.16):

$$\left(\nabla^2 + k^2 - \frac{(m-1)^2}{r^2\sin^2\theta} - \frac{2m(1-\cos\theta)}{r^2\sin^2\theta}\right)\tilde{a}^- = \qquad \text{(III.C.27)}$$

$$= -\frac{2}{r^2}\left(\frac{\partial}{\partial\theta} + \frac{m}{\sin\theta}\right)\left(ic_{lm}\, h_{l+1}^{(1)}(kr) P_l^m(\cos\theta)\right)$$

$$= 2ic_{lm}\frac{h_{l+1}^{(1)}(kr)}{r^2}\left(\frac{1-\cos\theta}{\sin^2\theta}\right)\frac{1}{2l+1} \,\cdot$$

$$\cdot \Big[l(l-m+1)(l-m+2)P_{l+1}^{m-1}(\cos\theta) - $$

$$- (2l+1)(l-m+1)(l+m)P_l^{m-1}(\cos\theta) + $$

$$+ (l+1)(l+m-1)(l+m)P_{l-1}^{m-1}(\cos\theta)\Big]$$

Assume a solution of the form

$$\tilde{a}^- = ic_{lm}h_{l+1}^{(1)}(kr)\left[b_1 P_{l+1}^{m-1}(\cos\theta) + b_2 P_l^{m-1}(\cos\theta)\right]$$

where the constants b_1 and b_2 are to be determined.

Plug this assumed form for $\tilde{a}^-$ into the L.H.S. of (III.C.27), and use equations (App.B2.1) and (App.C2.1) to calculate the effect of the ∇^2 operation:

$$\left(\nabla^2+k^2-\frac{(m-1)^2}{r^2\sin^2\theta}-\frac{2m(1-\cos\theta)}{r^2\sin^2\theta}\right)\tilde{a}^- =$$

$$= ic_{lm}\, b_1\left[\frac{(l+1)(l+2)-(l+1)((l+2)}{r^2} - \frac{2m(1-\cos\theta)}{r^2\sin^2\theta}\right]h^{(1)}_{l+1}(kr)P^{m-1}_{l+1}(\cos\theta)$$

$$+ ic_{lm}\, b_2\left[\frac{(l+1)(l+2)-l(l+1)}{r^2} - \frac{2m(1-\cos\theta)}{r^2\sin^2\theta}\right]h^{(1)}_{l+1}(kr)P^{m-1}_{l}(\cos\theta)$$

$$= 2ic_{lm}\frac{h^{(1)}_{l+1}}{r^2}\left(\frac{1-\cos\theta}{\sin^2\theta}\right)\Big[-b_1 mP^{m-1}_{l+1}(\cos\theta) +$$

$$+ b_2(l-m+1)P^{m-1}_{l}(\cos\theta) +$$

$$+ b_2(l+1)\cos\theta P^{m-1}_{l}(\cos\theta)\Big]$$

Use equation (App.C2.4) on the third term inside the brackets and pull out the (2l+1) term that gets introduced into the denominator:

$$= 2ic_{lm}\frac{h^{(1)}_{l+1}(kr)}{r^2}\left(\frac{1-\cos\theta}{\sin^2\theta}\right)\frac{1}{2l+1}\ \bullet$$

$$\bullet\ \Big[\Big(-b_1 m(2l+1) + b_2(l+1)(l-m+2)\Big)P^{m-1}_{l+1}(\cos\theta) -$$

$$- b_2(2l+1)(l-m+1)P^{m-1}_{l}(\cos\theta) +$$

$$+ b_2(l+1)(l+m-1)P^{m-1}_{l-1}(\cos\theta)\Big]$$

Equate the above expression to the R.H.S. of (III.C.27) to obtain coefficient matching conditions for b_1 and b_2:

$$-b_1 m(2l+1) + b_2(l+1)(l-m+2) = l(l-m+1)(l-m+2)$$

$$b_2(2l+1)(l+m+1) = (2l+1)(l-m+1)(l+m)$$

$$b_2(l+1)(l+m-1) = (l+1)(l+m-1)(l+m)$$

Solutions to the above trio of equations work out to be:

$$b_2 = (l+m) \qquad \text{(III.C.28)}$$

$$b_1 = (l-m+2) \qquad \text{(III.C.29)}$$

From which one obtains:

$$\tilde{a}^- = ic_{lm} h_{l+1}^{(1)}(kr)\Big[(l-m+2)P_{l+1}^{m-1}(\cos\theta) + (l+m)P_l^{m-1}(\cos\theta)\Big] \qquad \text{(III.C.30)}$$

D.) "General" solutions for ψ, a_r, a_θ, and a_ϕ

Two "particular" solutions for the $\tilde{a}^+$ equation have been derived, namely (III.C.18) and (III.C.22). General solutions are always obtainable from the set of particular solutions by combining them linearly in such a way that the sum of the expansion coefficients equals 1. In the case at hand, one would have:

$$\tilde{a}^+ = p_{lm}\left[-ic_{lm}h_{l+1}^{(1)}(kr)\left[\frac{1}{(l+m+1)}P_{l+1}^{m+1}(\cos\theta) + \frac{1}{(l+m+1)}P_l^{m+1}(\cos\theta)\right]\right]$$

$$+ q_{lm}\left[ic_{lm}h_{l+1}^{(1)}(kr)\left[-(l-m+2)P_{l+1}^{m-1}(\cos\theta) + (l+m)P_l^{m-1}(\cos\theta)\right]\right] \tag{III.D.1}$$

where p_{lm} and q_{lm} are any two constants such that:

$$p_{lm} + q_{lm} = 1$$

A convenient choice for p_{lm} and q_{lm} works out to be:

$$p_{lm} = \frac{1}{2}\frac{(l+m+1)}{(l+1)} + r_{lm}$$

$$q_{lm} = \frac{1}{2}\frac{(l-m+1)}{(l+1)} - r_{lm}$$

where r_{lm} is an arbitrary constant. (The motivation behind the above choice of p_{lm} and q_{lm} will be made clear in what follows.)

Plug these values of p_{lm} and q_{lm} into the (III.D.1) equation to obtain:

$$\tilde{a}^+ =$$

$$= \left\{ -\frac{i}{2}\frac{c_{lm}}{(l+1)} h^{(1)}_{l+1}(kr)\Big(P^{m+1}_{l+1}(\cos\theta) + P^{m+1}_{l}(\cos\theta)\Big) + \right.$$

$$\left. + \frac{i}{2}\frac{c_{lm}}{(l+1)} h^{(1)}_{l+1}(kr)(l-m+1)\Big(-(l-m+2)P^{m-1}_{l+1}(\cos\theta) + (l+m)P^{m-1}_{l}(\cos\theta)\Big)\right\}$$

$$- r_{lm}\left\{\frac{ic_{lm}}{(l+m+1)} h^{(1)}_{l+1}(kr)\Big(P^{m+1}_{l+1}(\cos\theta) + P^{m+1}_{l}(\cos\theta)\Big) + \right.$$

$$\left. + ic_{lm} h^{(1)}_{l+1}(kr)\Big(-(l-m+2)P^{m-1}_{l+1}(\cos\theta) + (l+m)P^{m-1}_{l}(\cos\theta)\Big)\right\}$$

Collect common factors and re-arrange terms within the brackets to obtain:

$$\tilde{a}^+ = -\frac{i}{2}\frac{c_{lm}}{(l+1)} h^{(1)}_{l+1}(kr)\left\{\Big(P^{m+1}_{l}(\cos\theta) - (l-m+1)(l+m)P^{m-1}_{l}(\cos\theta)\Big) + \right.$$

$$\left. + \Big(P^{m+1}_{l+1}(\cos\theta) + (l-m+1)(l-m+2)P^{m-1}_{l+1}(\cos\theta)\Big)\right\}$$

$$+ \frac{ir_{lm}\,c_{lm}}{(l+m+1)} h^{(1)}_{l+1}(kr)\left\{\Big(P^{m+1}_{l+1}(\cos\theta) - (l+m+1)(l-m+2)P^{m-1}_{l+1}(\cos\theta)\Big) + \right.$$

$$\left. + \Big(P^{m+1}_{l}(\cos\theta) + (l+m+1)(l+m)P^{m-1}_{l}(\cos\theta)\Big)\right\}$$

Define new expansion coefficients:

$$g_{lm} \equiv -\frac{c_{lm}}{(l+1)} \tag{III.D.2}$$

$$d_{(l+1)m} \equiv -\frac{r_{lm}\,c_{lm}}{(l+m+1)} \tag{III.D.3}$$

Then make use of the Legendre polynomial identities (App.C2.8,9 and 12) to obtain:

$$\begin{aligned}\tilde{a}^{+} &= i g_{lm}\, h^{(1)}_{l+1}(kr)\Big(\frac{dP_l^m(\cos\theta)}{d\theta} - \frac{m}{\sin\theta}P_l^m(\cos\theta)\Big)\\ &- i d_{(l+1)m}\, h^{(1)}_{l+1}(kr)\Big(\frac{dP_{l+1}^m(\cos\theta)}{d\theta} - \frac{m}{\sin\theta}P_{l+1}^m(\cos\theta)\Big)\end{aligned}$$

Adjust the index $(l+1) \to l$ on the second term to obtain:

$$\begin{aligned}\tilde{a}^{+} &= i g_{lm}\, h^{(1)}_{l+1}(kr)\Big(\frac{dP_l^m(\cos\theta)}{d\theta} - \frac{m}{\sin\theta}P_l^m(\cos\theta)\Big)\\ &- i d_{lm}\, h^{(1)}_{l}(kr)\Big(\frac{dP_l^m(\cos\theta)}{d\theta} - \frac{m}{\sin\theta}P_l^m(\cos\theta)\Big)\end{aligned} \tag{III.D.4}$$

The general solution for the $\tilde{a}^-$ equation proceeds analogously. Recall that two "particular" solutions for $\tilde{a}^-$ as given by (III.C.26) and (III.C.30) have been derived. The appropriate linear combination of "particular" solutions would be:

$$\tilde{a}^- = p_{lm}\Big[ic_{lm}h_{l+1}^{(1)}(kr)\Big[\frac{1}{(l+m+1)}P_{l+1}^{m+1}(\cos\theta) - \frac{1}{(l+m+1)}P_l^{m+1}(\cos\theta)\Big]\Big]$$

$$+ q_{lm}\Big[ic_{lm}h_{l+1}^{(1)}(kr)\Big[(l-m+2)P_{l+1}^{m-1}(\cos\theta) + (l+m)P_l^{m-1}(\cos\theta)\Big]\Big] \qquad \text{(III.D.5)}$$

where p_{lm} and q_{lm} are any two constants such that:

$$p_{lm} + q_{lm} = 1$$

The development from this point onward mimics the development for the $\tilde{a}^+$ solution provided in the previous paragraphs. Suffice it to simply state the final solution:

$$\tilde{a}^- = ig_{lm}h_{l+1}^{(1)}(kr)\Big(\frac{dP_l^m(\cos\theta)}{d\theta} + \frac{m}{\sin\theta}P_l^m(\cos\theta)\Big)$$

$$- id_{lm}h_l^{(1)}(kr)\Big(\frac{dP_l^m(\cos\theta)}{d\theta} + \frac{m}{\sin\theta}P_l^m(\cos\theta)\Big) \qquad \text{(III.D.6)}$$

Thus, general solutions for $\tilde{a}^+$ and $\tilde{a}^-$ have been derived. These solutions may now be employed to determine the components $\tilde{a}_\theta$ and $\tilde{a}_\phi$:

Recall from (III.C.10), (III.C.11):

$$\tilde{a}_\theta = \frac{1}{2}(\tilde{a}^+ + \tilde{a}^-)$$

$$\tilde{a}_\phi = \frac{1}{2i}(\tilde{a}^+ - \tilde{a}^-)$$

Utilizing (III.D.4) and (III.D.6), the $\tilde{a}_\theta$ and $\tilde{a}_\phi$ components become:

$$\tilde{a}_\theta = i g_{lm} h^{(1)}_{l+1}(kr)\frac{\partial P_l^m(\cos\theta)}{d\theta} + i d_{lm} h^{(1)}_l(kr)\frac{m}{\sin\theta}P_l^m(\cos\theta) \tag{III.D.7}$$

$$\tilde{a}_\phi = -g_{lm} h^{(1)}_{l+1}(kr)\frac{m}{\sin\theta}P_l^m(\cos\theta) - d_{lm} h^{(1)}_l(kr)\frac{dP_l^m(\cos\theta)}{d\theta} \tag{III.D.8}$$

Recalling (III.A.3), (III.C.7,8, and 14), and (III.D.2), one obtains general Helmholtz solutions:

$$\psi \;=\; -g_{lm}\,(l+1)\,h_l^{(1)}(kr)\,P_l^m(\cos\theta)\,e^{im\phi} \tag{III.D.9}$$

$$a_r \;=\; -i g_{lm}\,(l+1)\,h_{l+1}^{(1)}(kr)\,P_l^m(\cos\theta)\,e^{im\phi} \tag{III.D.10}$$

$$\begin{aligned} a_\theta \;=\;& \; i g_{lm}\, h_{l+1}^{(1)}(kr)\,\frac{dP_l^m(\cos\theta)}{d\theta}\,e^{im\phi} \\ &+\; i d_{lm}\, h_l^{(1)}(kr)\,\frac{m}{\sin\theta}\,P_l^m(\cos\theta)\,e^{im\phi} \end{aligned} \tag{III.D.11}$$

$$\begin{aligned} a_\phi \;=\;& -g_{lm}\, h_{l+1}^{(1)}(kr)\,\frac{m}{\sin\theta}\,P_l^m(\cos\theta)\,e^{im\phi} \\ &-\; d_{lm}\, h_l^{(1)}(kr)\,\frac{dP_l^m(\cos\theta)}{d\theta}\,e^{im\phi} \end{aligned} \tag{III.D.12}$$

Note the two independent expansion coefficients g_{lm} and d_{lm}.

The complete Helmholtz solution would be obtained by summing the above expressions over the range $l \;=\; (0 \text{ to } \infty)$ and $m \;=\; (-l \text{ to } +l)$.

E.) Coulomb Gauge

For problems of this sort, namely, in those regions of space where $(\rho, \vec{J}) = 0$, it is convenient to work in the Coulomb gauge where $\nabla \cdot \vec{a} = 0$.

This entails dividing $\vec{a}$ up into two separate components:

$$\vec{a} = \vec{a}^{(L)} + \vec{a}^{(T)} \tag{III.E.1}$$

where $\vec{a}^{(L)}$ and $\vec{a}^{(T)}$ are the "longitudinal" and "transverse" components of $\vec{a}$, respectively.

We require the divergence of $\vec{a}^{(T)}$ and the curl of $\vec{a}^{(L)}$ of equal zero.

Specifically:

$$\nabla \cdot \vec{a}^{(L)} = ik\psi \tag{III.E.2}$$

$$\nabla \times \vec{a}^{(L)} = \text{Zero} \tag{III.E.3}$$

$$\nabla \cdot \vec{a}^{(T)} = \text{Zero} \tag{III.E.4}$$

$$\nabla \times \vec{a}^{(T)} = \vec{b} \tag{III.E.5}$$

The objective is to define these transverse and longintudinal components of $\vec{a}$ such that the vector Helmholtz equation is still satisfied by each component individually:

$$\left(\nabla^2 + k^2\right)\vec{a}^{(L)} = \text{Zero} \tag{III.E.6}$$

$$\left(\nabla^2 + k^2\right)\vec{a}^{(T)} = \text{Zero} \tag{III.E.7}$$

This requirement can be achieved as follows:

$$\left(\nabla^2 + k^2\right)\vec{a}^{(L)} = \text{Zero}$$

$$\nabla\left(\nabla\cdot\vec{a}^{(L)}\right) - \nabla\times\nabla\times\vec{a}^{(L)} + k^2\vec{a}^{(L)} = \text{Zero}$$

$$\nabla\left(ik\psi\right) - \nabla\times(\text{Zero}) + k^2\vec{a}^{(L)} = \text{Zero}$$

$$k^2\vec{a}^{(L)} = -ik\nabla\psi$$

$$\vec{a}^{(L)} = -\frac{i}{k}\nabla\psi$$

Hence, given a general Helmholtzian $(\psi, \vec{a})$, one can obtain $\vec{a}^{(L)}$ and $\vec{a}^{(T)}$ as follows:

$$\vec{a}^{(L)} = -\frac{i}{k}\nabla\psi \qquad \text{(III.E.8)}$$

$$\vec{a}^{(T)} = \vec{a} - \vec{a}^{(L)} \qquad \text{(III.E.9)}$$

Notice:

$$\nabla\cdot\vec{a}^{(L)} = \nabla\cdot\left(-\frac{i}{k}\nabla\psi\right) = -\frac{i}{k}\nabla^2\psi = -\frac{i}{k}\left(-k^2\psi\right) = ik\psi \qquad \text{(III.E.10)}$$

And from (III.E.8):

$$\nabla\times\vec{a}^{(L)} = \text{Zero} \qquad \text{(III.E.11)}$$

$$ik\vec{a}^{(L)} = \nabla\psi \qquad \text{(III.E.12)}$$

Thus:

$$\vec{e} = -\nabla\psi + ik\vec{a}$$

$$= -\nabla\psi + ik(\vec{a}^{(L)} + \vec{a}^{(T)})$$

$$= ik\vec{a}^{(T)} \tag{III.E.13}$$

$$\vec{b} = \nabla\times\vec{a}$$

$$= \nabla\times(\vec{a}^{(L)} + \vec{a}^{(T)})$$

$$= \nabla\times\vec{a}^{(T)} \tag{III.E.14}$$

Hence, $\vec{e}$ and $\vec{b}$ are independent of the $\vec{a}^{(L)}$ component. The physics can be formulated entirely in terms of $\vec{a}^{(T)}$. In this situation, one is working in the "Coulomb" gauge because $\nabla\cdot\vec{a}^{(T)} = 0$.

Specifically:

$$(\nabla^2 + k^2)\vec{a}^{(T)} = \text{Zero} \tag{III.E.15}$$

$$\nabla\cdot\vec{a}^{(T)} = \text{Zero} \tag{III.E.16}$$

$$\vec{e} = ik\vec{a}^{(T)} \tag{III.E.17}$$

$$\vec{b} = \nabla\times\vec{a}^{(T)} \tag{III.E.18}$$

The objective now is to extract the transverse components $\vec{a}^{(T)}$ from the Helmholtzian $(\psi, \vec{a})$ solutions given by (III.D.9 thru 12) using the formulas (III.E.8) and (III.E.9).

One obtains Coulomb gauge solutions $\vec{a}^{(T)}$:

$$a_r^{(T)} = a_r + \frac{i}{k}\frac{\partial\psi}{\partial r} \tag{III.E.19}$$

$$= -ig_{lm}(l+1)\left(h_{l+1}^{(1)}(kr) + \frac{dh_l^{(1)}(kr)}{d(kr)}\right)P_l^m(\cos\theta)\,e^{im\phi}$$

$$= -ig_{lm}\,l(l+1)\frac{h_l^{(1)}(kr)}{kr}P_l^m(\cos\theta)\,e^{im\phi}$$

$$a_\theta^{(T)} = a_\theta + \frac{i}{kr}\frac{\partial\psi}{\partial\theta} \tag{III.E.20}$$

$$= ig_{lm}\left(h_{l+1}^{(1)}(kr) - (l+1)\frac{h_l^{(1)}(kr)}{kr}\right)\frac{dP_l^m(\cos\theta)}{d\theta}e^{im\phi}$$

$$+ \; id_{lm}\,h_l^{(1)}(kr)\frac{m}{\sin\theta}P_l^m(\cos\theta)\,e^{im\phi}$$

$$= -ig_{lm}\left(\frac{dh_l^{(1)}(kr)}{d(kr)} + \frac{h_l^{(1)}(kr)}{kr}\right)\frac{dP_l^m(\cos\theta)}{d\theta}e^{im\phi}$$

$$+ \; id_{lm}\,h_l^{(1)}(kr)\frac{m}{\sin\theta}P_l^m(\cos\theta)\,e^{im\phi}$$

$$a_\phi^{(T)} = a_\phi + \frac{i}{kr\sin\theta}\frac{\partial\psi}{\partial\phi} \tag{III.E.21}$$

$$= -g_{lm}\left(h_{l+1}^{(1)}(kr) - (l+1)h_l^{(1)}(kr)\right)\frac{m}{\sin\theta}P_l^m(\cos\theta)$$

$$- \; d_{lm}\,h_l^{(1)}(kr)\frac{dP_l^m(\cos\theta)}{d\theta}e^{im\phi}$$

$$= g_{lm}\left(\frac{dh_l^{(1)}(kr)}{d(kr)} + \frac{h_l^{(1)}(kr)}{kr}\right)\frac{m}{\sin\theta}P_l^m(\cos\theta)\,e^{im\phi}$$

$$- \; d_{lm}\,h_l^{(1)}(kr)\frac{dP_l^m(\cos\theta)}{d\theta}e^{im\phi}$$

Spherical Hankel identities (App.B2.14) and (App.B2.8) were utilized to re-express terms enclosed in the large parantheses in the above relations.

NOTE:

Early in this derivation, an optional expression was provided for a_r, namely equation (III.B.9):

$$a_r = -ic_{lm}h_{l-1}^{(1)}(kr)P_l^m(\cos\theta)e^{im\phi}$$

This option was then seemingly ignored.

As a matter of fact, had this option been selected, and then developed according to the program outlined in the Sections leading up to this one, $\bar{a}^{(T)}$ solutions identical to the one given above would have been obtained. It was therefore decided to forego reporting this parallel development.

F.) Maxwellian $\vec{E}$ and $\vec{B}$ solutions

One is now in a position to obtain expressions for $\vec{e}$, and $\vec{b}$. From (III.E.13):

$$\vec{e} = ik\vec{a}^{(T)}$$

and from (III.E.19 thru 21), one has:

$$e_r = g_{lm}\, l(l+1)\frac{h_l^{(1)}(kr)}{r} P_l^m(\cos\theta)\, e^{im\phi} \tag{III.F.1}$$

$$e_\theta = g_{lm}\left(\frac{dh_l^{(1)}(kr)}{dr} + \frac{h_l^{(1)}(kr)}{r}\right)\frac{dP_l^m(\cos\theta)}{d\theta}\, e^{im\phi} - k d_{lm}\, h_l^{(1)}(kr)\frac{m}{\sin\theta} P_l^m(\cos\theta)\, e^{im\phi} \tag{III.F.2}$$

$$e_\phi = i g_{lm}\left(\frac{dh_l^{(1)}(kr)}{dr} + \frac{h_l^{(1)}(kr)}{r}\right)\frac{m}{\sin\theta} P_l^m(\cos\theta)\, e^{im\phi} - ik d_{lm}\, h_l^{(1)}(kr)\frac{dP_l^m(\cos\theta)}{d\theta}\, e^{im\phi} \tag{III.F.3}$$

Likewise, from (III.E.14):

$$\vec{b} = \nabla \times \vec{a}^{(T)}$$

and from (III.E.19 thru 21), one has:

$$b_r = \frac{1}{r}\frac{\partial a_\phi^{(T)}}{\partial \theta} + \frac{\cos\theta}{r\sin\theta} a_\phi^{(T)} - \frac{1}{r\sin\theta}\frac{\partial a_\theta^{(T)}}{\partial \phi} \qquad \text{(III.F.4)}$$

$$= \frac{g_{lm}}{r}\left(\frac{dh_l^{(1)}(kr)}{d(kr)} + \frac{h_l^{(1)}(kr)}{kr}\right)\left(\frac{m}{\sin\theta}\frac{dP_l^m(\cos\theta)}{d\theta} - \frac{m\cos\theta}{\sin^2\theta}P_l^m(\cos\theta) + \right.$$

$$\left. + \frac{m\cos\theta}{\sin^2\theta}P_l^m(\cos\theta) - \frac{m}{\sin\theta}\frac{dP_l^m(\cos\theta)}{d\theta}\right)e^{im\phi}$$

$$- d_{lm}\frac{h_l^{(1)}(kr)}{r}\left(\frac{d^2P_l^m(\cos\theta)}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dP_l^m(\cos\theta)}{d\theta} - \frac{m^2}{\sin^2\theta}P_l^m(\cos\theta)\right)$$

$$= d_{lm}\, l(l+1)\frac{h_l^{(1)}(kr)}{r}P_l^m(\cos\theta)\, e^{im\phi}$$

$$b_\theta = \frac{1}{r\sin\theta}\frac{\partial a_r^{(T)}}{\partial\phi} - \frac{\partial a_\phi^{(T)}}{\partial r} - \frac{1}{r}a_\phi^{(T)} \qquad \text{(III.F.5)}$$

$$= \frac{g_{lm}}{k}\left(l(l+1)\frac{h_l^{(1)}(kr)}{r^2} - \frac{d^2h_l^{(1)}(kr)}{dr^2} - \frac{2}{r}\frac{dh_l^{(1)}(kr)}{dr}\right)\frac{m}{\sin\theta}P_l^m(\cos\theta)\,e^{im\phi}$$

$$+ \ d_{lm}\left(\frac{dh_l^{(1)}(kr)}{dr} + \frac{h_l^{(1)}(kr)}{r}\right)\frac{dP_l^m(\cos\theta)}{d\theta}e^{im\phi}$$

$$= d_{lm}\left(\frac{dh_l^{(1)}(kr)}{dr} + \frac{h_l^{(1)}(kr)}{r}\right)\frac{dP_l^m(\cos\theta)}{d\theta}e^{im\phi}$$

$$+ \ kg_{lm}h_l^{(1)}(kr)\frac{m}{\sin\theta}P_l^m(\cos\theta)\,e^{im\phi}$$

$$b_\phi = \frac{\partial a_\theta^{(T)}}{\partial r} + \frac{1}{r}a_\theta^{(T)} - \frac{1}{r}\frac{\partial a_r^{(T)}}{\partial\theta} \qquad \text{(III.F.6)}$$

$$= \frac{ig_{lm}}{k}\left(\frac{d^2h_l^{(1)}(kr)}{dr^2} + \frac{2}{r}\frac{dh_l^{(1)}(kr)}{dr} - l(l+1)\frac{h_l^{(1)}(kr)}{r^2}\right)\frac{dP_l^m(\cos\theta)}{d\theta}e^{im\phi}$$

$$+ \ id_{lm}\left(\frac{dh_l^{(1)}(kr)}{dr} + \frac{h_l^{(1)}(kr)}{r}\right)\frac{m}{\sin\theta}P_l^m(\cos\theta)\,e^{im\phi}$$

$$= id_{lm}\left(\frac{dh_l^{(1)}(kr)}{dr} + \frac{h_l^{(1)}(kr)}{r}\right)\frac{m}{\sin\theta}P_l^m(\cos\theta)\,e^{im\phi}$$

$$+ \ ikg_{lm}h_l^{(1)}(kr)\frac{dP_l^m(\cos\theta)}{d\theta}e^{im\phi}$$

Note the symmetry between the $\vec{e}$ and $\vec{b}$ solutions. The functional dependence upon (r, θ, ϕ) is the same for both vectors, the difference being that wherever expansion coefficients (g_{lm}, d_{lm}) appear in the $\vec{e}$ solution, $(d_{lm}, -g_{lm})$ appear in the $\vec{b}$ solution.

These g_{lm} and d_{lm} coefficients are (to within multiplicative constants) nothing other than the transverse electric and magnetic multipole moments, respectively, for the given system.

Also note that even though the above solutions have been derived in Coulomb gauge, the scalar potential ψ as given by (III.D.9) is *not* zero, despite frequently invoked claims to the contrary. The argument typically runs as follows. Since $\nabla \cdot \vec{a} = ik\psi$, and since in Coulomb gauge $\nabla \cdot \vec{a}^{(T)}$ equals zero by definition, then it "necessarily" follows that ψ is zero in this gauge.

The fallacy here is that $\vec{a}^{(T)} \neq \vec{a}$. (See equation III.E.9) The only restriction placed upon ψ is that it simultaneously satisfy the scalar Helmholtz equation (II.24) and the time-independent Lorentz condition (II.23).

The fact that ψ assumes non-zero values over space and time not only seems reasonable on intuitive grounds, but has ramifications in both theoretical and practical applications. Many problems of genuine physical interest cannot be properly solved if the scalar potential ψ is not utilized.

Using equations (II.8,9,14,15), (III.D.9 thru 12), and (III.F.1 thru 6), one obtains the final formulas for $\Phi(\vec{x},t)$, $\vec{A}(\vec{x},t)$, $\vec{E}(\vec{x},t)$, $\vec{B}(\vec{x},t)$:

$$\text{(III.F.7)} \qquad \Phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} -\frac{1}{2} g_{lm} (l+1) h_l^{(1)}(kr) P_l^m(\cos\theta) e^{im\phi} e^{-i\omega t} + c.c.$$

$$\text{(III.F.8)} \qquad A_r = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} -\frac{i}{2} g_{lm} (l+1) h_{l+1}^{(1)}(kr) P_l^m(\cos\theta) e^{im\phi} e^{-i\omega t} + c.c.$$

$$\text{(III.F.9)} \qquad A_\theta = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{i}{2} g_{lm} h_{l+1}^{(1)}(kr) \frac{dP_l^m(\cos\theta)}{d\theta} e^{im\phi} e^{-i\omega t} + \frac{i}{2} d_{lm} h_l^{(1)}(kr) \frac{m}{\sin\theta} P_l^m(\cos\theta) e^{im\phi} e^{-i\omega t} + c.c.$$

$$\text{(III.F.10)} \qquad A_\phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} -\frac{1}{2} g_{lm} h_{l+1}^{(1)}(kr) \frac{m}{\sin\theta} P_l^m(\cos\theta) e^{im\phi} e^{-i\omega t} - \frac{1}{2} d_{lm} h_l^{(1)}(kr) \frac{dP_l^m(\cos\theta)}{d\theta} e^{im\phi} e^{-i\omega t} + c.c.$$

$$\text{(III.F.11)} \qquad E_r = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2} g_{lm} l(l+1) \frac{h_l^{(1)}(kr)}{r} P_l^m(\cos\theta) e^{im\phi} e^{-i\omega t} + c.c.$$

$$\text{(III.F.12)} \qquad E_\theta = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2} g_{lm} \left(\frac{dh_l^{(1)}(kr)}{dr} + \frac{h_l^{(1)}(kr)}{r} \right) \frac{dP_l^m(\cos\theta)}{d\theta} e^{im\phi} e^{-i\omega t} - \frac{1}{2} k d_{lm} h_l^{(1)}(kr) \frac{m}{\sin\theta} P_l^m(\cos\theta) e^{im\phi} e^{-i\omega t} + c.c.$$

$$\text{(III.F.13)} \qquad E_\phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{i}{2} g_{lm} \left(\frac{dh_l^{(1)}(kr)}{dr} + \frac{h_l^{(1)}(kr)}{r} \right) \frac{m}{\sin\theta} P_l^m(\cos\theta) e^{im\phi} e^{-i\omega t} - \frac{i}{2} k d_{lm} h_l^{(1)}(kr) \frac{dP_l^m(\cos\theta)}{d\theta} e^{im\phi} e^{-i\omega t} + c.c.$$

(III.F.14) $$B_r = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2} d_{lm}\, l(l+1) \frac{h_l^{(1)}(kr)}{r} P_l^m(\cos\theta)\, e^{im\phi} e^{-i\omega t} + c.c.$$

(III.F.15) $$B_\theta = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2} d_{lm} \left(\frac{dh_l^{(1)}(kr)}{dr} + \frac{h_l^{(1)}(kr)}{r} \right) \frac{dP_l^m(\cos\theta)}{d\theta} e^{im\phi} e^{-i\omega t} + \frac{1}{2} k g_{lm}\, h_l^{(1)}(kr) \frac{m}{\sin\theta} P_l^m(\cos\theta)\, e^{im\phi} e^{-i\omega t} + c.c.$$

(III.F.16) $$B_\phi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{i}{2} d_{lm} \left(\frac{dh_l^{(1)}(kr)}{dr} + \frac{h_l^{(1)}(kr)}{r} \right) \frac{m}{\sin\theta} P_l^m(\cos\theta)\, e^{im\phi} e^{-i\omega t} + \frac{i}{2} k g_{lm}\, h_l^{(1)}(kr) \frac{dP_l^m(\cos\theta)}{d\theta} e^{im\phi} e^{-i\omega t} + c.c.$$

CHAPTER IV

SCALARS, VECTORS, TENSORS

A.) Transformation of Scalars, Vectors, and Tensors

The conversion from Cartesian components to spherical components is acheived via the transformation:

$$
\begin{aligned}
x &= r\sin\theta\cos\phi \\
y &= r\sin\theta\sin\phi \qquad \text{(IV.A.1)} \\
z &= r\cos\theta
\end{aligned}
$$

Using the notational aid of matrix multiplication, the familiar chain rule of partial differentiation for the operators $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$ can be stated in the succinct form:

$$
\begin{pmatrix} \dfrac{\partial\psi}{\partial x} \\ \dfrac{\partial\psi}{\partial y} \\ \dfrac{\partial\psi}{\partial z} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial\theta}{\partial x} & \dfrac{\partial\phi}{\partial x} \\ \dfrac{\partial r}{\partial y} & \dfrac{\partial\theta}{\partial y} & \dfrac{\partial\phi}{\partial y} \\ \dfrac{\partial r}{\partial z} & \dfrac{\partial\theta}{\partial z} & \dfrac{\partial\phi}{\partial z} \end{pmatrix} \begin{pmatrix} \dfrac{\partial\psi}{\partial r} \\ \dfrac{\partial\psi}{\partial\theta} \\ \dfrac{\partial\psi}{\partial\phi} \end{pmatrix} \qquad \text{(IV.A.2)}
$$

Using (IV.A.1) as a basis for calculation, and pulling common $1/r$ and

$1/r\sin\theta$ terms out from the matrix and into the R.H.S. columnn vector, one obtains:

$$\begin{pmatrix} \dfrac{\partial\psi}{\partial x} \\ \dfrac{\partial\psi}{\partial y} \\ \dfrac{\partial\psi}{\partial z} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial\psi}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial\psi}{\partial\phi} \end{pmatrix} \qquad \text{(IV.A.3)}$$

The spherical-to-Cartesian transformation matrix given on the R.H.S. of (IV.A.3) is pivotal. Note that the transpose of the given matrix is also its inverse. It completely characterizes the relationship between Cartesian and spherical quantities, and as such, will appear repeatedly in upcoming discussion. The tensor character of any quantity is uniquely determined by the number of times that this matrix (or its inverse) has to be used to to transform the given quantity from Cartesian coordinates to spherical coordinates. Scalar quantities do not require this matrix (or its inverse) in their transformation formulas (zeroth-order dependence). Vector quantities require its use one time in their transformation formulas (first-order dependence). Tensor quantitiies require its use two times in their transformation formulas (second-order dependence).

Specifically, a scalar is any quantity that transforms without the need of a transformation matrix:

$$\psi(\text{Cartesian}) = \psi(\text{spherical}) \tag{IV.A.4}$$

A vector is any ordered-combination of three quantities (V_x, V_y, V_z) that require the use of one matrix in their transformation formulas:

$$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} V_r \\ V_\theta \\ V_\phi \end{pmatrix} \tag{IV.A.5}$$

A tensor is any ordered-combination of nine quantities that require the use of two matrices in their transformation formulas:

$$\begin{pmatrix} T^{xx} & T^{xy} & T^{xz} \\ T^{yx} & T^{yy} & T^{yz} \\ T^{zx} & T^{zy} & T^{zz} \end{pmatrix} = \tag{IV.A.6}$$

$$= \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} T^{rr} & T^{r\theta} & T^{r\phi} \\ T^{\theta r} & T^{\theta\theta} & T^{\theta\phi} \\ T^{\phi r} & T^{\phi\theta} & T^{\phi\phi} \end{pmatrix} \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}$$

Scalar functions can be created from suitably-combined vector and tensor functions. A primary example is given by the dot product of two vectors (V_x, V_y, V_z) and (U_x, U_y, U_z) :

$$\vec{U}\cdot\vec{V} \;=\; (U_x, U_y, U_z)\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \;=\; (U_r, U_\theta, U_\phi)\begin{pmatrix} V_r \\ V_\theta \\ V_\phi \end{pmatrix} \qquad \text{(IV.A.7)}$$

In particular, if the vector $\vec{U}$ is taken equal to $\vec{V}$, the above would represent an expression for the *squared-norm* of $\vec{V}$.

Another important example of a scalar function is given by the sum of diagonal elements of any tensor. When the appropriate terms of (IV.A.6) are multiplied out and added, it is found that this combination of terms transforms without the need of a transformation matrix:

$$T^{xx} + T^{yy} + T^{zz} \;=\; T^{rr} + T^{\theta\theta} + T^{\phi\phi} \qquad \text{(IV.A.8)}$$

The above scalar quantity is denoted as the *trace* of any given tensor.

Similarly, vector functions can be constructed from scalar, vector, and tensor functions. An important example is the vector function constructed from the partial derivatives of a scalar function, (IV.A.3).

Another example is the vector function constructed from two other vector functions (V_x, V_y, V_z) and (U_x, U_y, U_z):

$$\begin{pmatrix} V_y V_z - V_z V_y \\ V_z V_x - V_x V_z \\ V_x V_y - V_y V_x \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} V_\theta V_\phi - V_\phi V_\theta \\ V_\phi V_r - V_r V_\phi \\ V_r V_\theta - V_\theta V_r \end{pmatrix}$$

(IV.A.9)

The above represents the vector *cross product* of two vectors.

Another example of a vector function is given by the subtracted combination of a tensor and its transpose. When the appropriate terms are subtracted, it is found that three independent components are obtained. When arrayed into a column and transformed according to the rules of (IV.A.6), it is found that these three components transform not as tensor, but rather as a vector:

$$\begin{pmatrix} (T^{yz} - T^{zy}) \\ (T^{zx} - T^{xz}) \\ (T^{xy} - T^{yx}) \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} (T^{\theta\phi} - T^{\phi\theta}) \\ (T^{\phi r} - T^{r\phi}) \\ (T^{r\theta} - T^{\theta r}) \end{pmatrix}$$

(IV.A.10)

The above relation underscores the basic *vector* nature of the *anti-symmetric* part of any tensor.

And lastly, tensor functions can be constructed from appropriately processed scalar and vector functions. The simplest example of a tensor constructed from a scalar would be:

$$\begin{pmatrix} \psi & 0 & 0 \\ 0 & \psi & 0 \\ 0 & 0 & \psi \end{pmatrix} = \qquad \text{(IV.A.11)}$$

$$= \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \psi & 0 & 0 \\ 0 & \psi & 0 \\ 0 & 0 & \psi \end{pmatrix} \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}$$

In particular, the above relation holds true for $\psi = \vec{U} \cdot \vec{V}$ or for $\psi = \mathit{Trace}\ \{T^{ij}\}$. This particular configuration is denoted the ***diagonal*** tensor.

Likewise, from the general vector (IV.A.5), one obtains:

$$\begin{pmatrix} 0 & -V_z & V_y \\ V_z & 0 & -V_x \\ -V_y & V_x & 0 \end{pmatrix} = \qquad \text{(IV.A.12)}$$

$$= \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} 0 & -V_\phi & V_\theta \\ V_\phi & 0 & -V_r \\ -V_\theta & V_r & 0 \end{pmatrix} \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}$$

The above is denoted the ***anti-symmetric*** tensor.

An example of a tensor function that can be constructed from two vectors (V_x, V_y, V_z) and (U_x, U_y, U_z) is:

$$\begin{pmatrix} V_x U_x & V_x U_y & V_x U_z \\ V_y U_x & V_y U_y & V_y U_z \\ V_z U_x & V_z U_y & V_z U_z \end{pmatrix} = \tag{IV.A.13}$$

$$= \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} V_r U_r & V_\theta U_\theta & V_r U_\phi \\ V_\theta U_r & V_\theta U_\theta & V_\theta U_\phi \\ V_\phi U_r & V_\phi U_\theta & V_\phi U_\phi \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}$$

The above is denoted the *dyadic* tensor.

One final word about tensors before proceeding to the next topic. From (IV.A.8), one observes that a one-element scalar component can always be extracted from a nine-element tensor array. From (IV.A.10), one observes that a three-element vector component can also be extracted. Thus, by elimination, only five elements remain to form the residual tensor once the scalar and vector components have been extracted. It is therefore possible to decompose the original nine-element tensor array into its one-element scalar, three-element vector, and five-element tensor components as follows:

$$\begin{pmatrix} T^{11} & T^{12} & T^{13} \\ T^{21} & T^{22} & T^{23} \\ T^{31} & T^{32} & T^{33} \end{pmatrix} = \tag{IV.A.14}$$

$$= \frac{1}{3}\begin{pmatrix} (T^{11}+T^{22}+T^{33}) & 0 & 0 \\ 0 & (T^{11}+T^{22}+T^{33}) & 0 \\ 0 & 0 & (T^{11}+T^{22}+T^{33}) \end{pmatrix}$$

$$+ \frac{1}{2}\begin{pmatrix} 0 & (T^{12}-T^{21}) & -(T^{31}-T^{13}) \\ -(T^{12}-T^{21}) & 0 & (T^{23}-T^{32}) \\ (T^{31}-T^{13}) & -(T^{23}-T^{32}) & 0 \end{pmatrix}$$

$$+ \frac{1}{2}\begin{pmatrix} \frac{2}{3}(2T^{11}-T^{22}-T^{33}) & (T^{12}+T^{21}) & (T^{31}+T^{13}) \\ (T^{12}+T^{21}) & \frac{2}{3}(-T^{11}+2T^{22}-T^{33}) & (T^{23}+T^{32}) \\ (T^{31}+T^{13}) & (T^{23}+T^{32}) & \frac{2}{3}(-T^{11}-T^{22}+2T^{33}) \end{pmatrix}$$

Although it may at first appear that there are six independent components in the third matrix (three on-diagonal elements and three independent off-diagonal elements), there are actually only five since the trace of the matrix is identically zero. Hence, the three elements along the diagonal are linear combinations of only two independent elements. There is some flexibility as to which two elements of the diagonal are to be taken as independent and which is to be dependent. With a view to upcoming applications in the spherical coordinate system, the following two terms are going to be assigned as the independent elements:

$$(T^{11}-T^{22}) \text{ and } (T^{22}-T^{33})$$

In terms of these two components, the symmeteric-traceless component of the original tensor array assumes the form:

$$\frac{1}{2}\begin{pmatrix} \frac{4}{3}(T^{11}-T^{22})+\frac{2}{3}(T^{22}-T^{33}) & (T^{12}+T^{21}) & (T^{31}+T^{13}) \\ (T^{12}+T^{21}) & -\frac{2}{3}(T^{11}-T^{22})+\frac{2}{3}(T^{22}-T^{33}) & (T^{23}+T^{32}) \\ (T^{31}+T^{13}) & (T^{23}+T^{32}) & -\frac{2}{3}(T^{11}-T^{22})-\frac{4}{3}(T^{22}-T^{33}) \end{pmatrix}$$

(IV.A.15)

Other investigators might devise different but equivalent forms for this matrix. But the one given above serves well in spherical coordinate applications, as will be demonstrated in upcoming sections of this report.

The tensor character of partial derivatives of vectors needs to be reviewed. Recall from earlier discussion that the ordered set of partial derivatives $\partial/\partial r$, $(1/r)\partial/\partial\theta$, and $(1/r\sin\theta)\partial/\partial\phi$ operating on a scalar function ψ yields a vector, as verified by the transformation law (IV.A.3). Since partial derivatives of a scalar produce a vector, it would be straightforward to assume that partial derivatives of a vector produce a tensor. Unfortunatey, things are not quite so simple in non-Cartesian coordinate systems. The curvilinear vector (V_r, V_θ, V_ϕ) contains coordinate-dependent pre-factors as stipulated by equation (IV.A.5). These pre-factors have to be included in the overall partial differentiations, and as such, complicate the final expressions. Utilizing (IV.A.2) and (IV.A.5), one quickly obtains:

$$\begin{pmatrix} \frac{\partial V_x}{\partial x} & \frac{\partial V_y}{\partial x} & \frac{\partial V_z}{\partial x} \\ \frac{\partial V_x}{\partial y} & \frac{\partial V_y}{\partial y} & \frac{\partial V_z}{\partial y} \\ \frac{\partial V_x}{\partial z} & \frac{\partial V_y}{\partial z} & \frac{\partial V_z}{\partial z} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \bullet \tag{IV.A.16}$$

$$\bullet \begin{pmatrix} \frac{\partial V_r}{\partial r} & \frac{\partial V_\theta}{\partial r} & \frac{\partial V_\phi}{\partial r} \\ \frac{1}{r}\frac{\partial V_r}{\partial\theta} - \frac{1}{r}V_\theta & \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{1}{r}V_r & \frac{1}{r}\frac{\partial V_\phi}{\partial\theta} \\ \frac{1}{r\sin\theta}\frac{\partial V_r}{\partial\phi} - \frac{1}{r}V_\phi & \frac{1}{r\sin\theta}\frac{\partial V_\theta}{\partial\phi} - \frac{\cos\theta}{r\sin\theta}V_\phi & \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi} + \frac{1}{r}V_r + \frac{\cos\theta}{r\sin\theta}V_\theta \end{pmatrix} \bullet$$

$$\bullet \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}$$

Note the appearance of extraneous terms alongside most of the spherical derivatives. Those familiar with tensor analysis will recognize these expressions as the "covariant derivatives" for the spherical system, frequently denoted $\partial V_m/\partial q^n - \Gamma^p_{mn} V_p$, where the indices (m,n,p) can assume values (r,θ,ϕ) and where summation over the index p is implied.

Extracting the trace of both sides of (IV.A.16) yields an expression for the scalar *Divergence* operation:

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z} = \frac{\partial V_r}{\partial r} + \frac{2}{r}V_r + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta} + \frac{\cos\theta}{r\sin\theta}V_\theta + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi} \tag{IV.A.17}$$

Extracting the anti-symmetric part of both sides and re-arranging terms into a column yields an expression for the vector *Curl* operation:

$$\begin{pmatrix} \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \\ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \\ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \cdot \tag{IV.A.18}$$

$$\cdot \begin{pmatrix} \frac{1}{r}\frac{\partial V_\phi}{\partial \theta} + \frac{\cos\theta}{r\sin\theta}V_\phi - \frac{1}{r\sin\theta}\frac{\partial V_\theta}{\partial \phi} \\ \frac{1}{r\sin\theta}\frac{\partial V_r}{\partial \phi} - \frac{\partial V_\phi}{\partial r} - \frac{1}{r}V_\phi \\ \frac{\partial V_\theta}{\partial r} + \frac{1}{r}V_\theta - \frac{1}{r}\frac{\partial V_r}{\partial \theta} \end{pmatrix}$$

Extracting the symmetric-traceless part of both sides yields a tensor comprised of five independent elements. Because of the large size of the matrices involved, the full equation will not fit onto one page.

(IV.A.19)

$$\begin{pmatrix} \frac{4}{3}\left(\frac{\partial V_x}{\partial x}-\frac{\partial V_y}{\partial y}\right)+\frac{2}{3}\left(\frac{\partial V_y}{\partial y}-\frac{\partial V_z}{\partial z}\right) & \left(\frac{\partial V_x}{\partial y}+\frac{\partial V_y}{\partial x}\right) & \left(\frac{\partial V_z}{\partial x}+\frac{\partial V_x}{\partial z}\right) \\ \left(\frac{\partial V_x}{\partial y}+\frac{\partial V_y}{\partial x}\right) & -\frac{2}{3}\left(\frac{\partial V_x}{\partial x}-\frac{\partial V_y}{\partial y}\right)+\frac{2}{3}\left(\frac{\partial V_y}{\partial y}-\frac{\partial V_z}{\partial z}\right) & \left(\frac{\partial V_y}{\partial z}+\frac{\partial V_z}{\partial y}\right) \\ \left(\frac{\partial V_z}{\partial x}+\frac{\partial V_x}{\partial z}\right) & \left(\frac{\partial V_y}{\partial z}+\frac{\partial V_z}{\partial y}\right) & -\frac{2}{3}\left(\frac{\partial V_x}{\partial x}-\frac{\partial V_y}{\partial y}\right)-\frac{4}{3}\left(\frac{\partial V_y}{\partial y}-\frac{\partial V_z}{\partial z}\right) \end{pmatrix} =$$

$$
= \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \cdot
$$

$$
\cdot \begin{pmatrix}
\frac{4}{3}\left(\frac{\partial V_r}{\partial r} - \frac{1}{r}V_r - \frac{1}{r}\frac{\partial V_\theta}{\partial \theta}\right) + \frac{2}{3}\left(\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} - \frac{\cos\theta}{r\sin\theta}V_\theta - \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi}\right) & \left(\frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{\partial V_\theta}{\partial r} - \frac{1}{r}V_\theta\right) & \left(\frac{1}{r\sin\theta}\frac{\partial V_r}{\partial \phi} + \frac{\partial V_\phi}{\partial r} - \frac{1}{r}V_\phi\right) \\
\left(\frac{1}{r}\frac{\partial V_r}{\partial \theta} + \frac{\partial V_\theta}{\partial r} - \frac{1}{r}V_\theta\right) & -\frac{2}{3}\left(\frac{\partial V_r}{\partial r} - \frac{1}{r}V_r - \frac{1}{r}\frac{\partial V_\theta}{\partial \theta}\right) + \frac{2}{3}\left(\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} - \frac{\cos\theta}{r\sin\theta}V_\theta - \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi}\right) & \left(\frac{1}{r\sin\theta}\frac{\partial V_\theta}{\partial \phi} + \frac{1}{r}\frac{\partial V_\phi}{\partial \theta} - \frac{\cos\theta}{r\sin\theta}V_\phi\right) \\
\left(\frac{1}{r\sin\theta}\frac{\partial V_r}{\partial \phi} + \frac{\partial V_\phi}{\partial r} - \frac{1}{r}V_\phi\right) & \left(\frac{1}{r\sin\theta}\frac{\partial V_\theta}{\partial \phi} + \frac{1}{r}\frac{\partial V_\phi}{\partial \theta} - \frac{\cos\theta}{r\sin\theta}V_\phi\right) & -\frac{2}{3}\left(\frac{\partial V_r}{\partial r} - \frac{1}{r}V_r - \frac{1}{r}\frac{\partial V_\theta}{\partial \theta}\right) - \frac{4}{3}\left(\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} - \frac{\cos\theta}{r\sin\theta}V_\theta - \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial \phi}\right)
\end{pmatrix} \cdot
$$

$$
\cdot \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix}
\tag{IV.A.19}
$$

B.) Transformation of Unit Vectors

The Cartesian unit vectors $(\hat{\imath},\hat{\jmath},\hat{k})$ and the spherical unit vectors $(\hat{r},\hat{\theta},\hat{\phi})$ are defined so as to linearly combine with vector functions such that the following equality is guaranteed:

$$\begin{pmatrix}\hat{r} & \hat{\theta} & \hat{\phi}\end{pmatrix}\begin{pmatrix}V_r \\ V_\theta \\ V_\phi\end{pmatrix} = \begin{pmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\end{pmatrix}\begin{pmatrix}V_x \\ V_y \\ V_z\end{pmatrix} \tag{IV.B.1}$$

Plugging (IV.A.5) into the R.H.S. yields:

$$\begin{pmatrix}\hat{r} & \hat{\theta} & \hat{\phi}\end{pmatrix}\begin{pmatrix}V_r \\ V_\theta \\ V_\phi\end{pmatrix} = \begin{pmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\end{pmatrix}\begin{pmatrix}\sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0\end{pmatrix}\begin{pmatrix}V_r \\ V_\theta \\ V_\phi\end{pmatrix} \tag{IV.B.2}$$

From which one obtains:

$$\begin{pmatrix}\hat{r} & \hat{\theta} & \hat{\phi}\end{pmatrix} = \begin{pmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\end{pmatrix}\begin{pmatrix}\sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0\end{pmatrix} \tag{IV.B.3}$$

Taking the transpose of the above and multiplying through by the appropriate inverse matrix puts it into (IV.A.5) format, thereby indicating that the ordered sets $(\hat{\imath},\hat{\jmath},\hat{k})$ and $(\hat{r},\hat{\theta},\hat{\phi})$ transform as vectors, as indeed they should. The inaugural statement for these unit vectors, (IV.B.1), is an example of (IV.A.7).

Partial derivatives of unit vectors play pivotal roles in many calculations. Since these derivatives are so fundamentally important in non-Cartesian formulations, it is beneficial to derive them in detail for the spherical system. Because there are three coordinates to consider, calculations will be done in stages and then consolidated at the end into one grand matrix.

First, one differentiates (IV.B.3) with respect to r:

$$\begin{pmatrix} \frac{\partial \hat{r}}{\partial r} & \frac{\partial \hat{\theta}}{\partial r} & \frac{\partial \hat{\phi}}{\partial r} \end{pmatrix} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \tag{IV.B.4}$$

Then one differentiates (IV.B.3) with respect to θ:

$$\begin{pmatrix} \frac{\partial \hat{r}}{\partial \theta} & \frac{\partial \hat{\theta}}{\partial \theta} & \frac{\partial \hat{\phi}}{\partial \theta} \end{pmatrix} = \tag{IV.B.5}$$

$$= \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \end{pmatrix} \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\cos\phi & 0 \\ \cos\theta\sin\phi & -\sin\theta\sin\phi & 0 \\ -\sin\theta & -\cos\theta & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \end{pmatrix} \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\cos\phi & 0 \\ \cos\theta\sin\phi & -\sin\theta\sin\phi & 0 \\ -\sin\theta & -\cos\theta & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \hat{\theta} & -\hat{r} & 0 \end{pmatrix}$$

Lastly, one differentiates (IV.B.3) with respect to ϕ:

$$\begin{pmatrix} \dfrac{\partial \hat{r}}{\partial \phi} & \dfrac{\partial \hat{\theta}}{\partial \phi} & \dfrac{\partial \hat{\phi}}{\partial \phi} \end{pmatrix} = \tag{IV.B.6}$$

$$= \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \end{pmatrix} \begin{pmatrix} -\sin\theta\sin\phi & -\cos\theta\sin\phi & -\cos\phi \\ \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \end{pmatrix} \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} -\sin\theta\sin\phi & -\cos\theta\sin\phi & -\cos\phi \\ \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \end{pmatrix} \begin{pmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & -\cos\theta \\ \sin\theta & \cos\theta & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sin\theta\,\hat{\phi} & \cos\theta\,\hat{\phi} & -(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}) \end{pmatrix}$$

These last three relations are incorporated into a single matrix expression:

$$\begin{pmatrix} \dfrac{\partial \hat{r}}{\partial r} & \dfrac{\partial \hat{\theta}}{\partial r} & \dfrac{\partial \hat{\phi}}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial \hat{r}}{\partial \theta} & \dfrac{1}{r}\dfrac{\partial \hat{\theta}}{\partial \theta} & \dfrac{1}{r}\dfrac{\partial \hat{\phi}}{\partial \theta} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial \hat{r}}{\partial \phi} & \dfrac{1}{r\sin\theta}\dfrac{\partial \hat{\theta}}{\partial \phi} & \dfrac{1}{r\sin\theta}\dfrac{\partial \hat{\phi}}{\partial \phi} \end{pmatrix} = \tag{IV.B.7}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ \dfrac{1}{r}\hat{\theta} & -\dfrac{1}{r}\hat{r} & 0 \\ \dfrac{1}{r}\hat{\phi} & \dfrac{\cos\theta}{r\sin\theta}\hat{\phi} & -\left(\dfrac{1}{r}\hat{r} + \dfrac{\cos\theta}{r\sin\theta}\hat{\theta}\right) \end{pmatrix}$$

Note an interesting thing here. If the vector $(\hat{r}, \hat{\theta}, \hat{\phi})$ is identified with the (V_r, V_θ, V_ϕ) of the R.H.S. of equation (IV.A.16), it is noted all nine elements of the (IV.A.16) array go to zero. Thus, all "covariant derivatives" of unit vectors vanish identically. This interesting property is exploited frequently in calculations involving vectors. It also serves as an alternate (and usually quicker) method of calculating covariant derivatives.

It only stands to reason that the R.H.S. matrix of (IV.A.16) should equal zero because the L.H.S. of this equation is nothing other than the statement that:

$$\begin{pmatrix} \frac{\partial \hat{\imath}}{\partial x} & \frac{\partial \hat{\jmath}}{\partial x} & \frac{\partial \hat{k}}{\partial x} \\ \frac{\partial \hat{\imath}}{\partial y} & \frac{\partial \hat{\jmath}}{\partial y} & \frac{\partial \hat{k}}{\partial y} \\ \frac{\partial \hat{\imath}}{\partial z} & \frac{\partial \hat{\jmath}}{\partial z} & \frac{\partial \hat{k}}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{IV.B.8}$$

These last two matrix relations epitomize the difference between the Cartesian and spherical coordinate systems. The non-zero behavior of the $\partial \hat{n} / \partial q_i$ terms in the spherical case has profound effects whenever working in this system. Consequences of this state of affairs are amply demonstrated in Chapter V.

C.) Transformation of Differential Operators

As was demonstrated in the previous section, the divergence operation can be expressed in covariant form:

(IV.C.1)

$$\nabla\cdot\vec{V} =$$

$$= \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \left(\left(\frac{\partial}{\partial r}+\frac{2}{r}\right),\left(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}\right),\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right)\begin{pmatrix} V_r \\ V_\theta \\ V_\phi \end{pmatrix}$$

Mindful of relations (IV.B.7) and (IV.B.8), the gradient and curl operations can also be expressed in covariant form. Each of these vector operations will be expressed in their covariant form (in both Cartesian and spherical systems), followed by their more familiar representation in component form (spherical system only).

The gradient is calculated using the diagonal tensor of (IV.A.11):

$$\nabla\psi = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\left[\begin{pmatrix} \psi & 0 & 0 \\ 0 & \psi & 0 \\ 0 & 0 & \psi \end{pmatrix}\begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix}\right] \qquad \text{(IV.C.2)}$$

$$= \left(\left(\frac{\partial}{\partial r}+\frac{2}{r}\right),\left(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}\right),\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right)\left[\begin{pmatrix} \psi & 0 & 0 \\ 0 & \psi & 0 \\ 0 & 0 & \psi \end{pmatrix}\begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix}\right]$$

$$= \hat{r}\frac{\partial\psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\ .$$

The curl is calculated using the anti-symmetric tensor of (IV.A.12):

$$\nabla \times \vec{V} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\left[\begin{pmatrix} 0 & -V_z & V_y \\ V_z & 0 & -V_x \\ -V_y & V_x & 0 \end{pmatrix}\begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix}\right] \quad \text{(IV.C.3)}$$

$$= \left(\left(\frac{\partial}{\partial r} + \frac{2}{r}\right), \left(\frac{1}{r}\frac{\partial}{\partial \theta} + \frac{\cos\theta}{r\sin\theta}\right), \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\right)\left[\begin{pmatrix} 0 & -V_\phi & V_\theta \\ V_\phi & 0 & -V_r \\ -V_\theta & V_r & 0 \end{pmatrix}\begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix}\right]$$

$$= \hat{r}\left(\frac{1}{r}\frac{\partial V_\phi}{\partial \theta} + \frac{\cos\theta}{r\sin\theta}V_\phi - \frac{1}{r\sin\theta}\frac{\partial V_\theta}{\partial \phi}\right)$$

$$+ \hat{\theta}\left(\frac{1}{r\sin\theta}\frac{\partial V_r}{\partial \phi} - \frac{\partial V_\phi}{\partial r} - \frac{1}{r}V_\phi\right)$$

$$+ \hat{\phi}\left(\frac{\partial V_\theta}{\partial r} + \frac{1}{r}V_\theta - \frac{1}{r}\frac{\partial V_r}{\partial \theta}\right)$$

The generalization of these formulas to *arbitrary* curvilinear coordinate systems should be obvious. (Simply replace subscripts and unit vectors as appropriate in the square-bracketted portions of the above.)

An important covariant differential operation involving the dyadic tensor (IV.A.13) is to come later, namely, (IV.C.8).

At this point, it becomes somewhat of a game to create covariant expressions for higher-order operations. For instance, the scalar Laplacian is given as:

(IV.C.4)

$$\begin{aligned}\nabla^2\psi &= \nabla\cdot\nabla\psi \\ &= \left(\left(\frac{\partial}{\partial r}+\frac{2}{r}\right), \left(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}\right), \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right)\begin{pmatrix}\dfrac{\partial\psi}{\partial r}\\ \dfrac{1}{r}\dfrac{\partial\psi}{\partial\theta}\\ \dfrac{1}{r\sin\theta}\dfrac{\partial\psi}{\partial\phi}\end{pmatrix}\end{aligned}$$

One can expand on the above by noting that $\vec{U}\cdot\vec{V}$ is a scalar and can therefore replace ψ wherever it appears in the formula:

$$\begin{aligned}\nabla^2(\vec{U}\cdot\vec{V}) &= \left(\left(\frac{\partial}{\partial r}+\frac{2}{r}\right), \left(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}\right), \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right)\cdot \\ &\quad\cdot\left[\begin{pmatrix}\dfrac{\partial}{\partial r}\\ \dfrac{1}{r}\dfrac{\partial}{\partial\theta}\\ \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\phi}\end{pmatrix}\left(U_r V_r + U_\theta V_\theta + U_\phi V_\phi\right)\right]\end{aligned} \qquad \text{(IV.C.5)}$$

Then, one lets $\vec{U} = (\hat{r}, \hat{\theta}, \hat{\phi})$ and $\vec{V} = (V_r, V_\theta, V_\phi)$ to obtain:

$$\nabla^2(\hat{r}\,V_r + \hat{\theta}\,V_\theta + \hat{\phi}\,V_\phi) = \tag{IV.C.6}$$

$$= \left(\left(\frac{\partial}{\partial r} + \frac{2}{r}\right), \left(\frac{1}{r}\frac{\partial}{\partial\theta} + \frac{\cos\theta}{r\sin\theta}\right), \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right) \right) \cdot$$

$$\cdot \left[\begin{pmatrix} \dfrac{\partial}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial}{\partial\theta} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\phi} \end{pmatrix} \left(\hat{r}\,V_r + \hat{\theta}\,V_\theta + \hat{\phi}\,V_\phi\right) \right]$$

Utilizing (IV.B.7), one obtains an explicit expression for the vector Laplacian in component form:

(IV.C.7)

$$\nabla^2\vec{V} = \hat{r}\left(\nabla^2 V_r - \frac{2}{r^2}V_r - \frac{2}{r^2}\frac{\partial V_\theta}{\partial\theta} - \frac{2\cos\theta}{r^2\sin\theta}V_\theta - \frac{2}{r^2\sin\theta}\frac{\partial V_\phi}{\partial\phi}\right)$$

$$+ \hat{\theta}\left(\nabla^2 V_\theta - \frac{1}{r^2\sin^2\theta}V_\theta + \frac{2}{r^2}\frac{\partial V_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial V_\phi}{\partial\phi}\right)$$

$$+ \hat{\phi}\left(\nabla^2 V_\phi - \frac{1}{r^2\sin^2\theta}V_\phi + \frac{2}{r^2\sin^2\theta}\frac{\partial V_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial V_\theta}{\partial\phi}\right)$$

Similar strategies can be used to obtain other covariant derivative expressions. One important example involves a symmetrized version of the dyadic tensor (IV.A.13). The particular expression given below, although not traceless, appears frequently in Chapter V of this report.

$$\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right) \cdot \tag{IV.C.8}$$

$$\cdot \left[\begin{pmatrix} V_x U_x + U_x V_x - \vec{V}\cdot\vec{U} & V_x U_y + U_x V_y & V_x U_z + U_x V_z \\ V_y U_x + U_y V_x & V_y U_y + U_y V_y - \vec{V}\cdot\vec{U} & V_y U_z + U_y V_z \\ V_z U_x + U_z V_x & V_z U_y + U_z V_y & V_z U_z + U_z V_z - \vec{V}\cdot\vec{U} \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix}\right]$$

$$= \left(\left(\frac{\partial}{\partial r}+\frac{2}{r}\right),\left(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}\right),\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right) \cdot$$

$$\cdot \left[\begin{pmatrix} V_r U_r + U_r V_r - \vec{V}\cdot\vec{U} & V_r U_\theta + U_r V_\theta & V_r U_\phi + U_r V_\phi \\ V_\theta U_r + U_\theta V_r & V_\theta U_\theta + U_\theta V_\theta - \vec{V}\cdot\vec{U} & V_\theta U_\phi + U_\theta V_\phi \\ V_\phi U_r + U_\phi V_r & V_\phi U_\theta + U_\phi V_\theta & V_\phi U_\phi + U_\phi V_\phi - \vec{V}\cdot\vec{U} \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix}\right]$$

$$= \vec{U}(\nabla\cdot\vec{V}) - \vec{V}\times(\nabla\times\vec{U}) + \vec{V}(\nabla\cdot\vec{U}) - \vec{U}\times(\nabla\times\vec{V})$$

Corollory:

$$\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\left[\begin{pmatrix} E_xE_x-\tfrac{1}{2}E^2 & E_xE_y & E_xE_z \\ E_yE_x & E_yE_y-\tfrac{1}{2}E^2 & E_yE_z \\ E_zE_x & E_zE_y & E_zE_z-\tfrac{1}{2}E^2 \end{pmatrix}\begin{pmatrix}\hat{\mathrm{i}}\\ \hat{\mathrm{j}}\\ \hat{\mathrm{k}}\end{pmatrix}\right] =$$

$$= \left(\left(\frac{\partial}{\partial r}+\frac{2}{r}\right),\left(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}\right),\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right)\cdot$$

$$\cdot\left[\begin{pmatrix} E_rE_r-\tfrac{1}{2}E^2 & E_rE_\theta & E_rE_\phi \\ E_\theta E_r & E_\theta E_\theta-\tfrac{1}{2}E^2 & E_\theta E_\phi \\ E_\phi E_r & E_\phi E_\theta & E_\phi E_\phi-\tfrac{1}{2}E^2 \end{pmatrix}\begin{pmatrix}\hat{\mathrm{r}}\\ \hat{\theta}\\ \hat{\phi}\end{pmatrix}\right] =$$

$$= \vec{\mathrm{E}}(\nabla\cdot\vec{\mathrm{E}})-\vec{\mathrm{E}}\times(\nabla\times\vec{\mathrm{E}}) \qquad \text{(IV.C.9)}$$

D.) Extension to 4-D Space

To fully exploit the covariant formalism, it is necessary to extend the mathematics of the previous sections to four dimensions. The unit vector $\hat{t}$ in the "time-direction" is considered independent of the spatial coordinates. Conversely, the spatial unit vectors $(\hat{i},\hat{j},\hat{k})$ are considered independent of the time coordinate.

One obtains natural extensions of Equations (IV.B.3,7,and 8) :

$$\begin{pmatrix} \hat{t} \\ \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ 0 & \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ 0 & \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \hat{t} \\ \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} \qquad \text{(IV.D.1)}$$

For Cartesian unit vectors:

$$\begin{pmatrix} \frac{1}{c}\frac{\partial \hat{t}}{\partial t} & \frac{1}{c}\frac{\partial \hat{i}}{\partial t} & \frac{1}{c}\frac{\partial \hat{j}}{\partial t} & \frac{1}{c}\frac{\partial \hat{k}}{\partial t} \\ \frac{\partial \hat{t}}{\partial x} & \frac{\partial \hat{i}}{\partial x} & \frac{\partial \hat{j}}{\partial x} & \frac{\partial \hat{k}}{\partial x} \\ \frac{\partial \hat{t}}{\partial y} & \frac{\partial \hat{i}}{\partial y} & \frac{\partial \hat{j}}{\partial y} & \frac{\partial \hat{k}}{\partial y} \\ \frac{\partial \hat{t}}{\partial z} & \frac{\partial \hat{i}}{\partial z} & \frac{\partial \hat{j}}{\partial z} & \frac{\partial \hat{k}}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{(IV.D.2)}$$

For spherical unit vectors:

(IV.D.3)

$$\begin{pmatrix} \frac{1}{c}\frac{\partial \hat{t}}{\partial t} & \frac{1}{c}\frac{\partial \hat{r}}{\partial t} & \frac{1}{c}\frac{\partial \hat{\theta}}{\partial t} & \frac{1}{c}\frac{\partial \hat{\phi}}{\partial t} \\ \frac{\partial \hat{t}}{\partial r} & \frac{\partial \hat{r}}{\partial r} & \frac{\partial \hat{\theta}}{\partial r} & \frac{\partial \hat{\phi}}{\partial r} \\ \frac{1}{r}\frac{\partial \hat{t}}{\partial \theta} & \frac{1}{r}\frac{\partial \hat{r}}{\partial \theta} & \frac{1}{r}\frac{\partial \hat{\theta}}{\partial \theta} & \frac{1}{r}\frac{\partial \hat{\phi}}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial \hat{t}}{\partial \phi} & \frac{1}{r\sin\theta}\frac{\partial \hat{r}}{\partial \phi} & \frac{1}{r\sin\theta}\frac{\partial \hat{\theta}}{\partial \phi} & \frac{1}{r\sin\theta}\frac{\partial \hat{\phi}}{\partial \phi} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{1}{r}\hat{\theta} & -\frac{1}{r}\hat{r} & 0 \\ 0 & \frac{1}{r}\hat{\phi} & \frac{\cos\theta}{r\sin\theta}\hat{\phi} & -\left(\frac{1}{r}\hat{r} + \frac{\cos\theta}{r\sin\theta}\hat{\theta}\right) \end{pmatrix}$$

The transformation laws for 4-scalars, 4-vectors, and 4-tensors are straightforward extensions of equations (IV.A.4), (IV.A.5) and (IV.A.6) :

Scalars:

$$\psi(\text{Cartesian}) = \psi(\text{spherical}) \tag{IV.D.4}$$

Vectors:

$$\begin{pmatrix} \Omega_t \\ \Omega_x \\ \Omega_y \\ \Omega_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ 0 & \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ 0 & \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \Omega_t \\ \Omega_r \\ \Omega_\theta \\ \Omega_\phi \end{pmatrix} \tag{IV.D.5}$$

Tensors:

$$\begin{pmatrix} \Theta_{tt} & \Theta_{tx} & \Theta_{ty} & \Theta_{tz} \\ \Theta_{xt} & \Theta_{xx} & \Theta_{xy} & \Theta_{xz} \\ \Theta_{yt} & \Theta_{yx} & \Theta_{yy} & \Theta_{yz} \\ \Theta_{zt} & \Theta_{zx} & \Theta_{zy} & \Theta_{zz} \end{pmatrix} = \tag{IV.D.6}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ 0 & \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ 0 & \cos\theta & -\sin\theta & 0 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} \Theta_{tt} & \Theta_{tr} & \Theta_{t\theta} & \Theta_{t\phi} \\ \Theta_{rt} & \Theta_{rr} & \Theta_{r\theta} & \Theta_{r\phi} \\ \Theta_{\theta t} & \Theta_{\theta r} & \Theta_{\theta\theta} & \Theta_{\theta\phi} \\ \Theta_{\phi t} & \Theta_{\phi r} & \Theta_{\phi\theta} & \Theta_{\phi\phi} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 0 & \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ 0 & -\sin\phi & \cos\phi & 0 \end{pmatrix}$$

An important distinction between 3-D and 4-D formulations involves vector length. In the 4-D formalism, one no longer works with a Euclidean metric, but rather with a Minkowskian metric. The dot product of two vectors is not a simple-minded extension of (IV.A.7), but is instead given as:

$$\Upsilon^i \Omega_i = \left(\Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3\right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \Omega_0 \\ \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} \qquad \text{(IV.D.7)}$$

$$= \Upsilon_0\Omega_0 - \Upsilon_1\Omega_1 - \Upsilon_2\Omega_2 - \Upsilon_3\Omega_3$$

The *squared-norm* of a vector in Minkowskian space would correspondingly be given as:

$$\Omega^i \Omega_i = \left(\Omega_0, \Omega_1, \Omega_2, \Omega_3\right) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \Omega_0 \\ \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} \qquad \text{(IV.D.8)}$$

$$= \Omega_0\Omega_0 - \Omega_1\Omega_1 - \Omega_2\Omega_2 - \Omega_3\Omega_3$$

Summation over the index i on the L.H.S. of the above two equations is understood. But note that this summation contains minus signs as indicated by the expressions on the R.H.S.'s. Without this particular selection of signs, the dot product of two 4-vectors does not properly contract down to a scalar invariant. A prime example of such an invariant is the squared-norm of the event coordinate (ct,x,y,z) , namely, $c^2t^2-x^2-y^2-z^2$.

The norm of a 4-vector in Minkowskian space is analogous to the length of a 3-vector in Euclidean space. In Euclidean space, one has the situation of a vector having components (x,y,z) in one system, and components (x',y',z') in another. Whatever system is chosen, the squared-length of the vector remains the same: $x^2+y^2+z^2 = x'^2+y'^2+z'^2$. Likewise, in the 4-D case, one would have that $c^2t^2-x^2-y^2-z^2 = c^2t'^2-x'^2-y'^2-z'^2$.

The diagonal matrix appearing in the R.H.S.'s of (IV.D.7) and (IV.D.8) is the Minkowskian metric and is denoted $\eta_{\mu\nu}$. This matrix must be utilized when contracting on any pair of vector or tensor indices.

The cumbersome effects of having to include matrix $\eta_{\mu\nu}$ in formulas can be relieved somewhat by defining contravariant and covariant vectors. Define the vector of (IV.D.5) as covariant. Note that the notation convention is to use subscripts on components of a covariant vector. The corresponding contravariant vector, whose components will be denoted with superscripts, is obtained by transposing the covariant vector from column to row format and multiplying through by the metric tensor $\eta_{\mu\nu}$:

Hence:

$$\left(\Omega^0,\Omega^1,\Omega^2,\Omega^3\right) = \left(\Omega_0,\Omega_1,\Omega_2,\Omega_3\right)\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} \qquad \text{(IV.D.9)}$$

$$= \left(\Omega_0,-\Omega_1,-\Omega_2,-\Omega_3\right)$$

Thus, the subscript-superscript notation used on the L.H.S.'s of (IV.D.7) and (IV.D.8) is now explained:

$$\Upsilon^i\Omega_i = \left(\Upsilon^0,\Upsilon^1,\Upsilon^2,\Upsilon^3\right)\begin{pmatrix}\Omega_0\\ \Omega_1\\ \Omega_2\\ \Omega_3\end{pmatrix} = \left(\Upsilon_0,-\Upsilon_1,-\Upsilon_2,-\Upsilon_3\right)\begin{pmatrix}\Omega_0\\ \Omega_1\\ \Omega_2\\ \Omega_3\end{pmatrix}$$

$$= \Upsilon_0\Omega_0 - \Upsilon_1\Omega_1 - \Upsilon_2\Omega_2 - \Upsilon_3\Omega_3 \qquad \text{(IV.D.10)}$$

This covariant-contravariant distinction also extends to 4-tensors. Consider the (IV.D.6) tensor to be doubly-covariant. Note that two sets of subscripts are used to describe the sixteen components. The corresponding co-contra, contra-co, and doubly-contravariant tensors work out to be:

$$\begin{pmatrix} \Theta_0{}^0 & \Theta_0{}^1 & \Theta_0{}^2 & \Theta_0{}^3 \\ \Theta_1{}^0 & \Theta_1{}^1 & \Theta_1{}^2 & \Theta_1{}^3 \\ \Theta_2{}^0 & \Theta_2{}^1 & \Theta_2{}^2 & \Theta_2{}^3 \\ \Theta_3{}^0 & \Theta_3{}^1 & \Theta_3{}^2 & \Theta_3{}^3 \end{pmatrix} = \begin{pmatrix} \Theta_{00} & \Theta_{01} & \Theta_{02} & \Theta_{03} \\ \Theta_{10} & \Theta_{11} & \Theta_{12} & \Theta_{13} \\ \Theta_{20} & \Theta_{21} & \Theta_{22} & \Theta_{23} \\ \Theta_{30} & \Theta_{31} & \Theta_{32} & \Theta_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{IV.D.11}$$

Likewise:

$$\begin{pmatrix} \Theta^0{}_0 & \Theta^0{}_1 & \Theta^0{}_2 & \Theta^0{}_3 \\ \Theta^1{}_0 & \Theta^1{}_1 & \Theta^1{}_2 & \Theta^1{}_3 \\ \Theta^2{}_0 & \Theta^2{}_1 & \Theta^2{}_2 & \Theta^2{}_3 \\ \Theta^3{}_0 & \Theta^3{}_1 & \Theta^3{}_2 & \Theta^3{}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \Theta_{00} & \Theta_{01} & \Theta_{02} & \Theta_{03} \\ \Theta_{10} & \Theta_{11} & \Theta_{12} & \Theta_{13} \\ \Theta_{20} & \Theta_{21} & \Theta_{22} & \Theta_{23} \\ \Theta_{30} & \Theta_{31} & \Theta_{32} & \Theta_{33} \end{pmatrix} \tag{IV.D.12}$$

Finally,

$$\begin{pmatrix} \Theta^{00} & \Theta^{01} & \Theta^{02} & \Theta^{03} \\ \Theta^{10} & \Theta^{11} & \Theta^{12} & \Theta^{13} \\ \Theta^{20} & \Theta^{21} & \Theta^{22} & \Theta^{23} \\ \Theta^{30} & \Theta^{31} & \Theta^{32} & \Theta^{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \Theta_{00} & \Theta_{01} & \Theta_{02} & \Theta_{03} \\ \Theta_{10} & \Theta_{11} & \Theta_{12} & \Theta_{13} \\ \Theta_{20} & \Theta_{21} & \Theta_{22} & \Theta_{23} \\ \Theta_{30} & \Theta_{31} & \Theta_{32} & \Theta_{33} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{IV.D.13}$$

Of the above four tensor types, it is only the two mixed-component tensors, (IV.D.11 and 12) whose traces form scalar invariants. The rule-of-thumb here is that contraction must always be performed over one contravariant index (notated as a supersript) and one covariant index (notated as a subscript). This combination of sub- and super- scripts occurs only in the two mixed-type tensors.

However, it is a very simple matter to obtain invariant scalars from the doubly-co and doubly-contra tensors. Simply extract a trace in the Minkowskian sense:

$$\text{Trace}\,\{T^{\mu\nu}\} = T^{00} - T^{11} - T^{22} - T^{33} \qquad \text{(IV.D.14)}$$

Since only the two unmixed-type tensors are used throughout the remainder of this report, it must be remembered that any "trace" calculation should be done in the above Minkowskian sense, that is, with one plus sign and three minus signs, to obtain a scalar quantity.

One final note about notation. In the 3-D formalism, there is no need to distinguish between covariant and contravariant components; hence, the subscript/superscript distinction holds no significance. Therefore, it will be my practice to simply continue using subscripts on 3-D vectors, regardless of their application. For example, (V_r, V_θ, V_ϕ) notation will be used even if this 3-vector is incorporated as part of a contravariant expression. Likewise for 3-tensors.

E.) Differential Operators in 4-D Space

The 4-D analog of the divergence operation is nothing more than an augmented version of equation (IV.C.1) :

$$\Box\cdot\boldsymbol{\Omega}_\alpha = \left(\frac{1}{c}\frac{\partial}{\partial t}\,,\,\frac{\partial}{\partial x}\,,\,\frac{\partial}{\partial y}\,,\,\frac{\partial}{\partial z}\right)\begin{pmatrix}\Omega_t\\ -\Omega_x\\ -\Omega_y\\ -\Omega_z\end{pmatrix} \qquad \text{(IV.E.1)}$$

$$= \left(\frac{1}{c}\frac{\partial}{\partial t}\,,\,\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\,,\,\left(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}\right)\,,\,\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right)\begin{pmatrix}\Omega_t\\ -\Omega_r\\ -\Omega_\theta\\ -\Omega_\phi\end{pmatrix}$$

$$= \frac{1}{c}\frac{\partial\Omega_t}{\partial t} - \nabla\cdot\vec{\Omega}$$

where the covariant 4-vector $\boldsymbol{\Omega}_\alpha$ is taken to be of the form $(\Omega_0,\Omega_1,\Omega_2,\Omega_3) = (\Omega_t,\vec{\Omega})$, where Ω_t and $\vec{\Omega}$ are, respectively, a scalar and a vector in the 3-D sense.

Note also the appearance of minus signs on the last three terms of the above covariant column vector. If a contravariant vector had been selected, and notated with superscripts rather than subscripts, there would be no need for minus signs on the last three components.

Equations (IV.A.11), (IV.A.12), and (IV.A.13), represent the diagonal, anti-symmetric, and dyadic tensors, respectively. Four dimensional analogs of these three tensor types that properly account for the Minkowskian nature of 4-space are easily constructed and then used to define the 4-D gradient, curl, and an operation that mimics equation (IV.C.8) .

The 4-D gradient is defined using the diagonal 4-tensor:

$$\Box\psi = \left(\frac{1}{c}\frac{\partial}{\partial t}\, ,\, \frac{\partial}{\partial x}\, ,\, \frac{\partial}{\partial y}\, ,\, \frac{\partial}{\partial z}\right)\left[\begin{pmatrix} \psi & 0 & 0 & 0 \\ 0 & -\psi & 0 & 0 \\ 0 & 0 & -\psi & 0 \\ 0 & 0 & 0 & -\psi \end{pmatrix}\begin{pmatrix} \hat{t} \\ \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix}\right] \tag{IV.E.2}$$

$$= \left(\frac{1}{c}\frac{\partial}{\partial t}\, ,\, \left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\, ,\, \left(\frac{1}{r}\frac{\partial}{\partial \theta}+\frac{\cos\theta}{r\sin\theta}\right)\, ,\, \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\right) \cdot$$

$$\cdot \left[\begin{pmatrix} \psi & 0 & 0 & 0 \\ 0 & -\psi & 0 & 0 \\ 0 & 0 & -\psi & 0 \\ 0 & 0 & 0 & -\psi \end{pmatrix}\begin{pmatrix} \hat{t} \\ \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix}\right]$$

$$= \hat{t}\,\frac{1}{c}\frac{\partial \psi}{\partial t} - \nabla\psi$$

The 4-D curl is defined using the anti-symmetric 4-tensor:

$$\Box \times \boldsymbol{\Omega}_{\alpha\beta} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \left[\begin{pmatrix} 0 & U_x & U_y & U_z \\ -U_x & 0 & V_z & -V_y \\ -U_y & -V_z & 0 & V_x \\ -U_z & V_y & -V_x & 0 \end{pmatrix} \begin{pmatrix} \hat{t} \\ \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix} \right] \qquad \text{(IV.E.3)}$$

$$= \left(\frac{1}{c}\frac{\partial}{\partial t}, \left(\frac{\partial}{\partial r} + \frac{2}{r}\right), \left(\frac{1}{r}\frac{\partial}{\partial \theta} + \frac{\cos\theta}{r\sin\theta}\right), \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\right) \bullet$$

$$\bullet \left[\begin{pmatrix} 0 & U_r & U_\theta & U_\phi \\ -U_r & 0 & V_\phi & -V_\theta \\ -U_\theta & -V_\phi & 0 & V_r \\ -U_\phi & V_\theta & -V_r & 0 \end{pmatrix} \begin{pmatrix} \hat{t} \\ \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix} \right]$$

$$= -\hat{t}\,\nabla\cdot\vec{U} + \frac{1}{c}\frac{\partial \vec{U}}{\partial t} - \nabla\times\vec{V}$$

The 4-D analog of equation (IV.C.8) is:

$$\left(\frac{1}{c}\frac{\partial}{\partial t}\,,\,\frac{\partial}{\partial x}\,,\,\frac{\partial}{\partial y}\,,\,\frac{\partial}{\partial z}\right)\,\cdot \tag{IV.E.4}$$

$$\cdot\left[\begin{pmatrix} \psi & W_x & W_y & W_z \\ W_x & V_xU_x+U_xV_x-\vec{V}\cdot\vec{U} & V_xU_y+U_xV_y & V_xU_z+U_xV_z \\ W_y & V_yU_x+U_yV_x & V_yU_y+U_yV_y-\vec{V}\cdot\vec{U} & V_yU_z+U_yV_z \\ W_z & V_zU_x+U_zV_x & V_zU_y+U_zV_y & V_zU_z+U_zV_z-\vec{V}\cdot\vec{U} \end{pmatrix}\begin{pmatrix}\hat{t}\\ \hat{i}\\ \hat{j}\\ \hat{k}\end{pmatrix}\right] =$$

$$= \left(\frac{1}{c}\frac{\partial}{\partial t}\,,\,\left(\frac{\partial}{\partial r}+\frac{2}{r}\right),\,\left(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}\right),\,\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right)\,\cdot$$

$$\cdot\left[\begin{pmatrix} \psi & W_r & W_\theta & W_\phi \\ W_r & V_rU_r+U_rV_r-\vec{V}\cdot\vec{U} & V_rU_\theta+U_rV_\theta & V_rU_\phi+U_rV_\phi \\ W_\theta & V_\theta U_r+U_\theta V_r & V_\theta U_\theta+U_\theta V_\theta-\vec{V}\cdot\vec{U} & V_\theta U_\phi+U_\theta V_\phi \\ W_\phi & V_\phi U_r+U_\phi V_r & V_\phi U_\theta+U_\phi V_\theta & V_\phi U_\phi+U_\phi V_\phi-\vec{V}\cdot\vec{U} \end{pmatrix}\begin{pmatrix}\hat{t}\\ \hat{r}\\ \hat{\theta}\\ \hat{\phi}\end{pmatrix}\right] =$$

$$= \hat{t}\left(\frac{1}{c}\frac{\partial\psi}{\partial t}+\nabla\cdot\vec{W}\right) + \frac{1}{c}\frac{\partial\vec{W}}{\partial t} + \vec{U}(\nabla\cdot\vec{V}) - \vec{V}\times(\nabla\times\vec{U}) + \vec{V}(\nabla\cdot\vec{U}) - \vec{U}\times(\nabla\times\vec{V})$$

Corollory:

In the previous formula, let :

$$\vec{U} = -(\vec{E} + i\vec{B})$$

$$\vec{V} = (\vec{E} - i\vec{B})$$

$$\vec{W} = (\vec{E} \times \vec{B})$$

$$\psi = ½(E^2 + B^2)$$

After the above plug-ins are made, the new matrices get unmanageably large. They therefore need to be split into two separate halves to fit onto a single page. The full equation, including both the Cartesian and spherical matrices and the final differential operator, extends onto the next two pages:

$$\left(\frac{1}{c}\frac{\partial}{\partial t}\,,\,\frac{\partial}{\partial x}\,,\,\frac{\partial}{\partial y}\,,\,\frac{\partial}{\partial z}\right)\cdot \tag{IV.E.5}$$

$$\cdot\left[\begin{pmatrix} \tfrac{1}{2}(\mathrm{E}^2+\mathrm{B}^2) & E_yB_z-E_zB_y & E_zB_x-E_xB_z & E_xB_y-E_yB_x \\ E_yB_z-E_zB_y & -E_xE_x-B_xB_x+\tfrac{1}{2}(\mathrm{E}^2+\mathrm{B}^2) & -E_xE_y-B_xB_y & -E_xE_z-B_xB_z \\ E_zB_x-E_xB_z & -E_yE_x-B_yB_x & -E_yE_y-B_yB_y+\tfrac{1}{2}(\mathrm{E}^2+\mathrm{B}^2) & -E_yE_z-B_yB_z \\ E_xB_y-E_yB_x & -E_zE_x-B_zB_x & -E_zE_y-B_zB_y & -E_zE_z-B_zB_z+\tfrac{1}{2}(\mathrm{E}^2+\mathrm{B}^2) \end{pmatrix}\begin{pmatrix}\hat{\mathrm{t}}\\ \hat{\mathrm{i}}\\ \hat{\mathrm{j}}\\ \hat{\mathrm{k}}\end{pmatrix}\right] =$$

$$= \left(\frac{1}{c}\frac{\partial}{\partial t}\,,\,\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\,,\,\left(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}\right)\,,\,\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right)\cdot$$

$$\cdot\left[\begin{pmatrix} \tfrac{1}{2}(\mathrm{E}^2+\mathrm{B}^2) & E_\theta B_\phi-E_\phi B_\theta & E_\phi B_r-E_rB_\phi & E_rB_\theta-E_\theta B_r \\ E_\theta B_\phi-E_\phi B_\theta & -E_rE_r-B_rB_r+\tfrac{1}{2}(\mathrm{E}^2+\mathrm{B}^2) & -E_rE_\theta-B_rB_\theta & -E_rE_\phi-B_rB_\phi \\ E_\phi B_r-E_rB_\phi & -E_\theta E_r-B_\theta B_r & -E_\theta E_\theta-B_\theta B_\theta+\tfrac{1}{2}(\mathrm{E}^2+\mathrm{B}^2) & -E_\theta E_\phi-B_\theta B_\phi \\ E_rB_\theta-E_\theta B_r & -E_\phi E_r-B_\phi B_r & -E_\phi E_\theta-B_\phi B_\theta & -E_\phi E_\phi-B_\phi B_\phi+\tfrac{1}{2}(\mathrm{E}^2+\mathrm{B}^2) \end{pmatrix}\begin{pmatrix}\hat{\mathrm{t}}\\ \hat{\mathrm{r}}\\ \hat{\theta}\\ \hat{\phi}\end{pmatrix}\right] =$$

$$= \hat{t}\Big(\frac{1}{2c}\frac{\partial}{\partial t}(E^2+B^2) + \nabla\cdot(\vec{E}\times\vec{B})\Big) +$$

$$+ \frac{1}{c}\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \vec{E}(\nabla\cdot\vec{E}) + \vec{E}\times(\nabla\times\vec{E}) - \vec{B}(\nabla\cdot\vec{B}) + \vec{B}\times(\nabla\times\vec{B})$$

The use of a few vector identities on the above expression yields:

$$= \hat{t}\Big(\vec{E}\cdot\frac{1}{c}\frac{\partial\vec{E}}{\partial t} + \vec{B}\cdot\frac{1}{c}\frac{\partial\vec{B}}{\partial t} + \vec{B}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{B})\Big) +$$

$$+ \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\times\vec{B} + \vec{E}\times\frac{1}{c}\frac{\partial\vec{B}}{\partial t} - \vec{E}(\nabla\cdot\vec{E}) + \vec{E}\times(\nabla\times\vec{E}) - \vec{B}(\nabla\cdot\vec{B}) + \vec{B}\times(\nabla\times\vec{B})$$

Collecting terms yields:

$$= \hat{t}\Big(\vec{B}\cdot\big(\nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t}\big) - \vec{E}\cdot\big(\nabla\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\big)\Big) -$$

$$- \vec{E}(\nabla\cdot\vec{E}) - \vec{B}(\nabla\cdot\vec{B}) + \vec{E}\times\big(\nabla\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t}\big) + \vec{B}\times\big(\nabla\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\big)$$

Recognizing that all terms enclosed in smaller-sized parantheses are expressible in terms of ρ and $\vec{J}$ by virtue of Maxwell's equations (II.1), one final equality may be made:

$$= -\frac{4\pi}{c}\Big[\hat{t}\big(\vec{J}\cdot\vec{E}\big) + c\rho\vec{E} + \vec{J}\times\vec{B}\Big] \qquad \text{(IV.E.5)}$$

To conclude this section, it is necessary to augment the 3-D formulas (IV.A.14) through (IV.A.19). First, we have the 4-D equivalent of (IV.A.14/15). It will be shown below that the sixteen-element doubly-covariant 4-tensor can be decomposed into a one-element scalar part, a six-element vector part, and a nine-element tensor part. Since all work is done in a Minkowskian 4-space, the signature combination of one positive and three negative terms in trace expressions will be recurrently observed.

Also of note here is the particular linear combination of terms used to construct the on-diagonal elements of the nine-element tensor part. These three elements are never uniquely defined, but whatever the choice, they must identically form a Minkowskian sum of zero. My own selection is presented in the third matrix of the upcoming equation; others might devise different combinations. My particular choice happens to have advantages when the tensor is transformed to spherical coordinates, as will be demonstrated later.

The full equation extends over two pages. We have:

$$\begin{pmatrix} T_{00} & T_{01} & T_{02} & T_{03} \\ T_{10} & T_{11} & T_{12} & T_{13} \\ T_{20} & T_{21} & T_{22} & T_{23} \\ T_{30} & T_{31} & T_{32} & T_{33} \end{pmatrix} = \tag{IV.E.6}$$

$$= \frac{1}{4} \begin{pmatrix} (T_{00}-T_{11}-T_{22}-T_{33}) & 0 & 0 & 0 \\ 0 & -(T_{00}-T_{11}-T_{22}-T_{33}) & 0 & 0 \\ 0 & 0 & -(T_{00}-T_{11}-T_{22}-T_{33}) & 0 \\ 0 & 0 & 0 & -(T_{00}-T_{11}-T_{22}-T_{33}) \end{pmatrix} +$$

$$+ \frac{1}{2} \begin{pmatrix} 0 & -(T_{10}-T_{01}) & -(T_{20}-T_{02}) & -(T_{30}-T_{03}) \\ (T_{10}-T_{01}) & 0 & (T_{12}-T_{21}) & -(T_{31}-T_{13}) \\ (T_{20}-T_{02}) & -(T_{12}-T_{21}) & 0 & (T_{23}-T_{32}) \\ (T_{30}-T_{03}) & (T_{31}-T_{13}) & -(T_{23}-T_{32}) & 0 \end{pmatrix} +$$

$$+\ \frac{1}{2}\begin{pmatrix} \frac{3}{2}(T_{00}+T_{11})-(T_{11}-T_{22})-\frac{1}{2}(T_{22}-T_{33}) & (T_{10}+T_{01}) & (T_{20}+T_{02}) & (T_{30}+T_{03}) \\ (T_{10}+T_{01}) & \frac{1}{2}(T_{00}+T_{11})+(T_{11}-T_{22})+\frac{1}{2}(T_{22}-T_{33}) & (T_{12}+T_{21}) & (T_{31}+T_{13}) \\ (T_{20}+T_{02}) & (T_{12}+T_{21}) & \frac{1}{2}(T_{00}+T_{11})-(T_{11}-T_{22})+\frac{1}{2}(T_{22}-T_{33}) & (T_{23}+T_{32}) \\ (T_{30}+T_{03}) & (T_{31}+T_{13}) & (T_{23}+T_{32}) & \frac{1}{2}(T_{00}+T_{11})-(T_{11}-T_{22})-\frac{3}{2}(T_{22}-T_{33}) \end{pmatrix}$$

(IV.E.6)

Next, we have the 4-D equivalent of (IV.A.16):

$$\begin{pmatrix} \frac{1}{c}\frac{\partial\Omega_t}{\partial t} & \frac{1}{c}\frac{\partial\Omega_x}{\partial t} & \frac{1}{c}\frac{\partial\Omega_y}{\partial t} & \frac{1}{c}\frac{\partial\Omega_z}{\partial t} \\ \frac{\partial\Omega_t}{\partial x} & \frac{\partial\Omega_x}{\partial x} & \frac{\partial\Omega_y}{\partial x} & \frac{\partial\Omega_z}{\partial x} \\ \frac{\partial\Omega_t}{\partial y} & \frac{\partial\Omega_x}{\partial y} & \frac{\partial\Omega_y}{\partial y} & \frac{\partial\Omega_z}{\partial y} \\ \frac{\partial\Omega_t}{\partial z} & \frac{\partial\Omega_x}{\partial z} & \frac{\partial\Omega_y}{\partial z} & \frac{\partial\Omega_z}{\partial z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ 0 & \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ 0 & \cos\theta & -\sin\theta & 0 \end{pmatrix} \cdot$$

$$\begin{pmatrix} \frac{1}{c}\frac{\partial\Omega_t}{\partial t} & \frac{1}{c}\frac{\partial\Omega_r}{\partial t} & \frac{1}{c}\frac{\partial\Omega_\theta}{\partial t} & \frac{1}{c}\frac{\partial\Omega_\phi}{\partial t} \\ \frac{\partial\Omega_t}{\partial r} & \frac{\partial\Omega_r}{\partial r} & \frac{\partial\Omega_\theta}{\partial r} & \frac{\partial\Omega_\phi}{\partial r} \\ \frac{1}{r}\frac{\partial\Omega_t}{\partial\theta} & \frac{1}{r}\frac{\partial\Omega_r}{\partial\theta}-\frac{1}{r}\Omega_\theta & \frac{1}{r}\frac{\partial\Omega_\theta}{\partial\theta}+\frac{1}{r}\Omega_r & \frac{1}{r}\frac{\partial\Omega_\phi}{\partial\theta} \\ \frac{1}{r\sin\theta}\frac{\partial\Omega_t}{\partial\phi} & \frac{1}{r\sin\theta}\frac{\partial\Omega_r}{\partial\phi}-\frac{1}{r}\Omega_\phi & \frac{1}{r\sin\theta}\frac{\partial\Omega_\theta}{\partial\phi}-\frac{\cos\theta}{r\sin\theta}\Omega_\phi & \frac{1}{r\sin\theta}\frac{\partial\Omega_\phi}{\partial\phi}+\frac{1}{r}\Omega_r+\frac{\cos\theta}{r\sin\theta}\Omega_\theta \end{pmatrix}$$

$$\cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 0 & \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ 0 & -\sin\phi & \cos\phi & 0 \end{pmatrix} \qquad \text{(IV.E.7)}$$

It will prove informative to decompose (IV.E.7) according to the prescriptions of (IV.E.6). Because of space limitations, this will be done only for the spherical side of the equation. Thus, the sixteen-element array on the R.H.S. of (IV.E.7) may be re-expressed:

(IV.E.8)

$$
= \frac{1}{4}\begin{pmatrix}
\left(\frac{1}{c}\frac{\partial \Omega_t}{\partial t}-\nabla\cdot\bar{\Omega}\right) & 0 & 0 & 0 \\
0 & -\left(\frac{1}{c}\frac{\partial \Omega_t}{\partial t}-\nabla\cdot\bar{\Omega}\right) & 0 & 0 \\
0 & 0 & -\left(\frac{1}{c}\frac{\partial \Omega_t}{\partial t}-\nabla\cdot\bar{\Omega}\right) & 0 \\
0 & 0 & 0 & -\left(\frac{1}{c}\frac{\partial \Omega_t}{\partial t}-\nabla\cdot\bar{\Omega}\right)
\end{pmatrix} +
$$

$$
+ \frac{1}{2}\begin{pmatrix}
0 & \left(\frac{1}{c}\frac{\partial \Omega_r}{\partial t}-\frac{\partial \Omega_t}{\partial r}\right) & \left(\frac{1}{c}\frac{\partial \Omega_\theta}{\partial t}-\frac{1}{r}\frac{\partial \Omega_t}{\partial \theta}\right) & \left(\frac{1}{c}\frac{\partial \Omega_\phi}{\partial t}-\frac{1}{r\sin\theta}\frac{\partial \Omega_t}{\partial \phi}\right) \\
-\left(\frac{1}{c}\frac{\partial \Omega_r}{\partial t}-\frac{\partial \Omega_t}{\partial r}\right) & 0 & \left(\frac{\partial \Omega_\theta}{\partial r}+\frac{1}{r}\Omega_\theta-\frac{1}{r}\frac{\partial \Omega_r}{\partial \theta}\right) & -\left(\frac{1}{r\sin\theta}\frac{\partial \Omega_r}{\partial \phi}-\frac{\partial \Omega_\phi}{\partial r}-\frac{1}{r}\Omega_\phi\right) \\
-\left(\frac{1}{c}\frac{\partial \Omega_\theta}{\partial t}-\frac{1}{r}\frac{\partial \Omega_t}{\partial \theta}\right) & -\left(\frac{\partial \Omega_\theta}{\partial r}+\frac{1}{r}\Omega_\theta-\frac{1}{r}\frac{\partial \Omega_r}{\partial \theta}\right) & 0 & \left(\frac{1}{r}\frac{\partial \Omega_\phi}{\partial \theta}+\frac{\cos\theta}{r\sin\theta}\Omega_\phi-\frac{1}{r\sin\theta}\frac{\partial \Omega_\theta}{\partial \phi}\right) \\
-\left(\frac{1}{c}\frac{\partial \Omega_\phi}{\partial t}-\frac{1}{r\sin\theta}\frac{\partial \Omega_t}{\partial \phi}\right) & \left(\frac{1}{r\sin\theta}\frac{\partial \Omega_r}{\partial \phi}-\frac{\partial \Omega_\phi}{\partial r}-\frac{1}{r}\Omega_\phi\right) & -\left(\frac{1}{r}\frac{\partial \Omega_\phi}{\partial \theta}+\frac{\cos\theta}{r\sin\theta}\Omega_\phi-\frac{1}{r\sin\theta}\frac{\partial \Omega_\theta}{\partial \phi}\right) & 0
\end{pmatrix} +
$$

$$
+\ \frac{1}{2}\begin{pmatrix}
\frac{3}{2}\left(\frac{1}{c}\frac{\partial\Omega_t}{\partial t}+\frac{\partial\Omega_r}{\partial r}\right)-\left(\frac{\partial\Omega_r}{\partial r}-\frac{1}{r}\Omega_r-\frac{1}{r}\frac{\partial\Omega_\theta}{\partial\theta}\right)-\frac{1}{2}\left(\frac{1}{r}\frac{\partial\Omega_\theta}{\partial\theta}-\frac{\cos\theta}{r\sin\theta}\Omega_\theta-\frac{1}{r\sin\theta}\frac{\partial\Omega_\phi}{\partial\phi}\right) & \left(\frac{\partial\Omega_t}{\partial r}+\frac{1}{c}\frac{\partial\Omega_r}{\partial t}\right) & \left(\frac{1}{r}\frac{\partial\Omega_t}{\partial\theta}+\frac{1}{c}\frac{\partial\Omega_\theta}{\partial t}\right) & \left(\frac{1}{r\sin\theta}\frac{\partial\Omega_t}{\partial t}+\frac{1}{c}\frac{\partial\Omega_\phi}{\partial t}\right) \\
\left(\frac{\partial\Omega_t}{\partial r}+\frac{1}{c}\frac{\partial\Omega_r}{\partial t}\right) & \frac{1}{2}\left(\frac{1}{c}\frac{\partial\Omega_t}{\partial t}+\frac{\partial\Omega_r}{\partial r}\right)+\left(\frac{\partial\Omega_r}{\partial r}-\frac{1}{r}\Omega_r-\frac{1}{r}\frac{\partial\Omega_\theta}{\partial\theta}\right)+\frac{1}{2}\left(\frac{1}{r}\frac{\partial\Omega_\theta}{\partial\theta}-\frac{\cos\theta}{r\sin\theta}\Omega_\theta-\frac{1}{r\sin\theta}\frac{\partial\Omega_\phi}{\partial\phi}\right) & \left(\frac{\partial\Omega_\theta}{\partial r}-\frac{1}{r}\Omega_\theta+\frac{1}{r}\frac{\partial\Omega_r}{\partial\theta}\right) & \left(\frac{1}{r\sin\theta}\frac{\partial\Omega_r}{\partial\phi}+\frac{\partial\Omega_\phi}{\partial r}-\frac{1}{r}\Omega_\phi\right) \\
\left(\frac{1}{r}\frac{\partial\Omega_t}{\partial\theta}+\frac{1}{c}\frac{\partial\Omega_\theta}{\partial t}\right) & \left(\frac{\partial\Omega_\theta}{\partial r}-\frac{1}{r}\Omega_\theta+\frac{1}{r}\frac{\partial\Omega_r}{\partial\theta}\right) & \frac{1}{2}\left(\frac{1}{c}\frac{\partial\Omega_t}{\partial t}+\frac{\partial\Omega_r}{\partial r}\right)-\left(\frac{\partial\Omega_r}{\partial r}-\frac{1}{r}\Omega_r-\frac{1}{r}\frac{\partial\Omega_\theta}{\partial\theta}\right)+\frac{1}{2}\left(\frac{1}{r}\frac{\partial\Omega_\theta}{\partial\theta}-\frac{\cos\theta}{r\sin\theta}\Omega_\theta-\frac{1}{r\sin\theta}\frac{\partial\Omega_\phi}{\partial\phi}\right) & \left(\frac{1}{r}\frac{\partial\Omega_\phi}{\partial\theta}-\frac{\cos\theta}{r\sin\theta}\Omega_\phi+\frac{1}{r\sin\theta}\frac{\partial\Omega_\theta}{\partial\phi}\right) \\
\left(\frac{1}{r\sin\theta}\frac{\partial\Omega_t}{\partial t}+\frac{1}{c}\frac{\partial\Omega_\phi}{\partial t}\right) & \left(\frac{1}{r\sin\theta}\frac{\partial\Omega_r}{\partial\phi}+\frac{\partial\Omega_\phi}{\partial r}-\frac{1}{r}\Omega_\phi\right) & \left(\frac{1}{r}\frac{\partial\Omega_\phi}{\partial\theta}-\frac{\cos\theta}{r\sin\theta}\Omega_\phi+\frac{1}{r\sin\theta}\frac{\partial\Omega_\theta}{\partial\phi}\right) & \frac{1}{2}\left(\frac{1}{c}\frac{\partial\Omega_t}{\partial t}+\frac{\partial\Omega_r}{\partial r}\right)-\left(\frac{\partial\Omega_r}{\partial r}-\frac{1}{r}\Omega_r-\frac{1}{r}\frac{\partial\Omega_\theta}{\partial\theta}\right)-\frac{3}{2}\left(\frac{1}{r}\frac{\partial\Omega_\theta}{\partial\theta}-\frac{\cos\theta}{r\sin\theta}\Omega_\theta-\frac{1}{r\sin\theta}\frac{\partial\Omega_\phi}{\partial\phi}\right)
\end{pmatrix}
$$

(IV.E.8)

There is only one independent element in the first matrix, namely, the scalar 4-divergence as given by (IV.E.1):

$$\frac{1}{c}\frac{\partial \Omega_t}{\partial t} - \nabla\cdot\vec{\Omega} \tag{IV.E.9}$$

There are six independent elements in the second matrix, arrayed so as to form an anti-symmetric 4x4. There is a one-to-one correspondence between the six elements of this second matrix and the general antisymmetric matrix as given in the bracketted portion of (IV.E.3). From this correspondence, it is easy to see that two 3-vectors are extractable from this anti-symmetric matrix:

$$\begin{pmatrix} \dfrac{1}{r}\dfrac{\partial \Omega_\phi}{\partial \theta} + \dfrac{\cos\theta}{r\sin\theta}\Omega_\phi - \dfrac{1}{r\sin\theta}\dfrac{\partial \Omega_\theta}{\partial \phi} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial \Omega_r}{\partial \phi} - \dfrac{\partial \Omega_\phi}{\partial r} - \dfrac{1}{r}\Omega_\phi \\ \dfrac{\partial \Omega_\theta}{\partial r} + \dfrac{1}{r}\Omega_\theta - \dfrac{1}{r}\dfrac{\partial \Omega_r}{\partial \theta} \end{pmatrix} \tag{IV.E.10}$$

And:

$$\begin{pmatrix} \dfrac{1}{c}\dfrac{\partial \Omega_r}{\partial t} - \dfrac{\partial \Omega_t}{\partial r} \\ \dfrac{1}{c}\dfrac{\partial \Omega_\theta}{\partial t} - \dfrac{1}{r}\dfrac{\partial \Omega_t}{\partial \theta} \\ \dfrac{1}{c}\dfrac{\partial \Omega_\phi}{\partial t} - \dfrac{1}{r\sin\theta}\dfrac{\partial \Omega_t}{\partial \phi} \end{pmatrix} \tag{IV.E.11}$$

The first of the above vectors should be recognized as the spherical representation of the 3-D curl as specified in (IV.C.3). The second of the two vectors should be recognized as $\frac{1}{c}\frac{\partial\vec{\Omega}}{\partial t} - \nabla\Omega_t$. Contrast this expression with (IV.E.9).

There are nine independent elements in the third matrix, arrayed so as to form a symmetric-traceless 4x4. ("Traceless" taken in the Minkowskian sense, IV.D.14). The six off-diagonal elements are unambiguously defined from the original tensor, (IV.E.7). The three on-diagonal elements are not uniquely defined, as has been discussed previously. The particular selection utilized here assures that each of the three on-diagonal elements contains two, and *only* two, spherical components Ω_i and Ω_j. This maneuver allows for some mathematical simplifications later on.

CHAPTER V

DERIVATION OF CONSERVED ELECTROMAGNETIC QUANTITIES

A.) Electrodynamics in 4-D Covariant Form

Now that the 4-D formalism has been developed, it is a straightforward exercise to apply it to Maxwellian electrodynamics.

The first 4-vector quantity that we have at our disposal is the 4-D charge-current density :

$$(c\rho, \vec{\mathrm{J}}) \equiv \mathbf{J}^{\alpha} \tag{V.A.1}$$

The continuity equation for electric charge, equation (I.2), tells us that the 4-Divergence of the above contravariant quantity is zero:

$$0 \;=\; \Box\cdot(c\rho, \vec{\mathrm{J}}) \;=\; \Box\cdot\mathbf{J}^{\alpha} \tag{V.A.2}$$

$$= \left(\frac{1}{c}\frac{\partial}{\partial t}\,,\,\frac{\partial}{\partial x}\,,\,\frac{\partial}{\partial y}\,,\,\frac{\partial}{\partial z}\right)\begin{pmatrix} c\rho \\ J^{x} \\ J^{y} \\ J^{z} \end{pmatrix}$$

$$= \left(\frac{1}{c}\frac{\partial}{\partial t}\,,\,\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\,,\,\left(\frac{1}{r}\frac{\partial}{\partial \theta}+\frac{\cos\theta}{r\sin\theta}\right)\,,\,\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\right)\begin{pmatrix} c\rho \\ J^{r} \\ J^{\theta} \\ J^{\phi} \end{pmatrix}$$

Any vector whose divergence is zero must be expressible as a curl. In the 4-D case, this means:

$$(c\rho, \vec{J}) = \Big(\frac{1}{c}\frac{\partial}{\partial t}\,,\,\big(\frac{\partial}{\partial r}+\frac{2}{r}\big)\,,\,\big(\frac{1}{r}\frac{\partial}{\partial \theta}+\frac{\cos\theta}{r\sin\theta}\big)\,,\,\big(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\big)\Big) \,\cdot$$

$$\cdot \left[\begin{pmatrix} 0 & U_r & U_\theta & U_\phi \\ -U_r & 0 & V_\phi & -V_\theta \\ -U_\theta & -V_\phi & 0 & V_r \\ -U_\phi & V_\theta & -V_r & 0 \end{pmatrix}\begin{pmatrix} \hat{t} \\ \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix}\right]$$

$$= \; -\hat{t}\,\nabla\cdot\vec{U} \;+\; \frac{1}{c}\frac{\partial \vec{U}}{\partial t} \;-\; \nabla\times\vec{V} \qquad \text{(V.A.3)}$$

Comparison of the above curl formula with the two inhomogeneous Maxwell equations, (II.1a) and (II.1d), reveals immediately the identity of the vectors $\vec{U}$ and $\vec{V}$:

$$\vec{U} = -\frac{c}{4\pi}\vec{E}$$

$$\vec{V} = -\frac{c}{4\pi}\vec{B}$$

Hence, the two inhomogeneous Maxwell equations expressed in the 4-D formalism become:

$$(c\rho, \vec{J}) = \left(\frac{1}{c}\frac{\partial}{\partial t}\,,\,\left(\frac{\partial}{\partial r}+\frac{2}{r}\right),\,\left(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}\right),\,\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right) \bullet \qquad \text{(V.A.4)}$$

$$\bullet \left[\frac{c}{4\pi}\begin{pmatrix} 0 & -E_r & -E_\theta & -E_\phi \\ E_r & 0 & -B_\phi & B_\theta \\ E_\theta & B_\phi & 0 & -B_r \\ E_\phi & -B_\theta & B_r & 0 \end{pmatrix}\begin{pmatrix} \hat{t} \\ \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix}\right] .$$

What is interesting here is that (V.A.2) *forces* the existence of an equation of the form (V.A.4) . There is no arbitrariness in the form of the two inhomogeneous Maxwell equations.

The two homogenous Maxwell equations, (II.1b) and (II.1c), fall out analogously:

$$\text{Zero} = \left(\frac{1}{c}\frac{\partial}{\partial t}\,,\,\left(\frac{\partial}{\partial r}+\frac{2}{r}\right),\,\left(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}\right),\,\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right) \bullet \qquad \text{(V.A.5)}$$

$$\bullet \left[\frac{c}{4\pi}\begin{pmatrix} 0 & B_r & B_\theta & B_\phi \\ -B_r & 0 & -E_\phi & E_\theta \\ -B_\theta & E_\phi & 0 & -E_r \\ -B_\phi & -E_\theta & E_r & 0 \end{pmatrix}\begin{pmatrix} \hat{t} \\ \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix}\right]$$

Attention is next turned to the electromagnetic quantities u, $\vec{g}$ and $\vec{\sigma}$, the (scalar) energy, (vector) momentum, and (tensor) stress densities, respectively. The equations governing these densities are almost as important as Maxwell's equations themselves.

Definitions of these densities in terms of the electric and magnetic fields are provided in virtually all E&M texts:

$$u = \frac{1}{8\pi}\left(E^2+B^2\right) \tag{V.A.6}$$

$$\vec{g} = \frac{1}{4\pi c}\left(\vec{E}\times\vec{B}\right) \tag{V.A.7}$$

$$\sigma_{ij} = \frac{1}{4\pi}\left(-E_iE_j - B_iB_j + \tfrac{1}{2}\delta_{ij}(E^2+B^2)\right) \tag{V.A.8}$$

The tensor-differential equation that inter-relates all three of these electromagnetic densities is (IV.E.5). After multiplying through by $1/4\pi$, identification of the various terms in the two tensor arrays becomes obvious.

$$\Box\cdot\left[\begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix}\begin{pmatrix} \hat{\mathrm{n}}_0 \\ \hat{\mathrm{n}}_1 \\ \hat{\mathrm{n}}_2 \\ \hat{\mathrm{n}}_3 \end{pmatrix}\right] = \qquad \text{(V.A.9)}$$

$$= \left(\frac{1}{c}\frac{\partial}{\partial t}\,,\,\frac{\partial}{\partial x}\,,\,\frac{\partial}{\partial y}\,,\,\frac{\partial}{\partial z}\right)\left[\begin{pmatrix} u & cg_x & cg_y & cg_z \\ cg_x & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ cg_y & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ cg_z & \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}\begin{pmatrix} \hat{\mathrm{t}} \\ \hat{\mathrm{i}} \\ \hat{\mathrm{j}} \\ \hat{\mathrm{k}} \end{pmatrix}\right] =$$

$$= \left(\frac{1}{c}\frac{\partial}{\partial t}\,,\,\left(\frac{\partial}{\partial r}+\frac{2}{r}\right),\,\left(\frac{1}{r}\frac{\partial}{\partial \theta}+\frac{\cos\theta}{r\sin\theta}\right),\,\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\right)\,\cdot$$

$$\cdot\left[\begin{pmatrix} u & cg_r & cg_\theta & cg_\phi \\ cg_r & \sigma_{rr} & \sigma_{r\theta} & \sigma_{r\phi} \\ cg_\theta & \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta\phi} \\ cg_\phi & \sigma_{\phi r} & \sigma_{\phi\theta} & \sigma_{\phi\phi} \end{pmatrix}\begin{pmatrix} \hat{\mathrm{t}} \\ \hat{\mathrm{r}} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix}\right] =$$

$$= -\frac{1}{c}\left[\hat{\mathrm{t}}(\vec{\mathrm{J}}\cdot\vec{\mathrm{E}}) + c\rho\vec{\mathrm{E}} + \vec{\mathrm{J}}\times\vec{\mathrm{B}}\right]$$

For those source-free regions of space where $(c\rho, \vec{\mathrm{J}}) = 0$, the L.H.S.'s of equations (V.A.4) and (V.A.5) and the R.H.S. of equation (V.A.9) equal zero. In this special case, certain mathematical properties become operative, consequences of which are explored in the upcoming section.

B.) Extracting Divergenceless Quantities from Covariant Expressions

From the previous section, it is clear that the equations of electrodynamics in source-free space repeatedly reduce to tensor relations of the form:

$$\text{Zero} \;=\; \left(\frac{1}{c}\frac{\partial}{\partial t}\,,\,\left(\frac{\partial}{\partial r}+\frac{2}{r}\right),\,\left(\frac{1}{r}\frac{\partial}{\partial \theta}+\frac{\cos\theta}{r\sin\theta}\right),\,\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\right)\cdot \left[\begin{pmatrix} T^{tt} & T^{tr} & T^{t\theta} & T^{t\phi} \\ T^{rt} & T^{rr} & T^{r\theta} & T^{r\phi} \\ T^{\theta t} & T^{\theta r} & T^{\theta\theta} & T^{\theta\phi} \\ T^{\phi t} & T^{\phi r} & T^{\phi\theta} & T^{\phi\phi} \end{pmatrix}\begin{pmatrix} \hat{t} \\ \hat{r} \\ \hat{\theta} \\ \hat{\phi} \end{pmatrix}\right] \qquad \text{(V.B.1)}$$

Expanding the above expression, one obtains:

$$\begin{aligned} \text{Zero} \;=\; & \frac{1}{c}\frac{\partial}{\partial t}\left(\hat{t}\,T^{tt}+\hat{r}\,T^{tr}+\hat{\theta}\,T^{t\theta}+\hat{\phi}\,T^{t\phi}\right) \\ & +\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\left(\hat{t}\,T^{rt}+\hat{r}\,T^{rr}+\hat{\theta}\,T^{r\theta}+\hat{\phi}\,T^{r\phi}\right) \\ & +\left(\frac{1}{r}\frac{\partial}{\partial \theta}+\frac{\cos\theta}{r\sin\theta}\right)\left(\hat{t}\,T^{\theta t}+\hat{r}\,T^{\theta r}+\hat{\theta}\,T^{\theta\theta}+\hat{\phi}\,T^{\theta\phi}\right) \\ & +\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\left(\hat{t}\,T^{\phi t}+\hat{r}\,T^{\phi r}+\hat{\theta}\,T^{\phi\theta}+\hat{\phi}\,T^{\phi\phi}\right) \end{aligned} \qquad \text{(V.B.2)}$$

Mindful that partial derivatives of non-Cartesian unit vectors are *not necessarily zero,* the above expression must be expanded out according to the prescriptions of (IV.D.3). Then, by setting the coefficients of the four $\hat{t}, \hat{r}, \hat{\theta}, \hat{\phi}$ components to zero, one obtains:

$$\frac{1}{c}\frac{\partial T^{tt}}{\partial t} + \left(\frac{\partial}{\partial r}+\frac{2}{r}\right)T^{rt} + \left(\frac{1}{r}\frac{\partial}{\partial \theta}+\frac{\cos\theta}{r\sin\theta}\right)T^{\theta t} + \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)T^{\phi t} =$$

$$= \text{Zero} \qquad \text{(V.B.3)}$$

$$\frac{1}{c}\frac{\partial T^{tr}}{\partial t} + \left(\frac{\partial}{\partial r}+\frac{2}{r}\right)T^{rr} + \left(\frac{1}{r}\frac{\partial}{\partial \theta}+\frac{\cos\theta}{r\sin\theta}\right)T^{\theta r} + \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)T^{\phi r} =$$

$$= \frac{1}{r}T^{\theta\theta} + \frac{1}{r}T^{\phi\phi} \qquad \text{(V.B.4)}$$

$$\frac{1}{c}\frac{\partial T^{t\theta}}{\partial t} + \left(\frac{\partial}{\partial r}+\frac{2}{r}\right)T^{r\theta} + \left(\frac{1}{r}\frac{\partial}{\partial \theta}+\frac{\cos\theta}{r\sin\theta}\right)T^{\theta\theta} + \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)T^{\phi\theta} =$$

$$= -\frac{1}{r}T^{\theta r} + \frac{\cos\theta}{r\sin\theta}T^{\phi\phi} \qquad \text{(V.B.5)}$$

$$\frac{1}{c}\frac{\partial T^{t\phi}}{\partial t} + \left(\frac{\partial}{\partial r}+\frac{2}{r}\right)T^{r\phi} + \left(\frac{1}{r}\frac{\partial}{\partial \theta}+\frac{\cos\theta}{r\sin\theta}\right)T^{\theta\phi} + \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)T^{\phi\phi} =$$

$$= -\frac{1}{r}T^{\phi r} - \frac{\cos\theta}{r\sin\theta}T^{\phi\theta} \qquad \text{(V.B.6)}$$

Any differential relation of the form:

$$\frac{1}{c}\frac{\partial a^0}{\partial t}+\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)a^1+\left(\frac{1}{r}\frac{\partial}{\partial \theta}+\frac{\cos\theta}{r\sin\theta}\right)a^2+\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\right)a^3 = \text{Zero}$$

represents a *conservation law* for the quantity $\int a^0 dV$. Of the four equations given above, only (V.B.3) falls into this special category.

The objective therefore is to extract more conservation laws by converting the above system of inhomogeneous P.D.E.'s into an equivalent set of homogeneous P.D.E.'s. The method is to linearly combine the four equations in appropriate ways such that the residual terms on the R.H.S.'s of the above equations exactly cancel one another out.

Mathematically, we seek four functions:

$$\begin{aligned} F &= F(t,r,\theta,\phi) \\ G &= G(t,r,\theta,\phi) \\ H &= H(t,r,\theta,\phi) \\ I &= I(t,r,\theta,\phi) \end{aligned} \tag{V.B.7}$$

such that:

$$\text{Zero} = \left(\frac{1}{c}\frac{\partial}{\partial t}\,,\,\left(\frac{\partial}{\partial r}+\frac{2}{r}\right),\,\left(\frac{1}{r}\frac{\partial}{\partial \theta}+\frac{\cos\theta}{r\sin\theta}\right),\,\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right)\cdot \left[\begin{pmatrix} T^{tt} & T^{tr} & T^{t\theta} & T^{t\phi} \\ T^{rt} & T^{rr} & T^{r\theta} & T^{r\phi} \\ T^{\theta t} & T^{\theta r} & T^{\theta\theta} & T^{\theta\phi} \\ T^{\phi t} & T^{\phi r} & T^{\phi\theta} & T^{\phi\phi} \end{pmatrix}\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix}\right] \tag{V.B.8}$$

$$\begin{aligned} = {} & \frac{1}{c}\frac{\partial}{\partial t}\left(F T^{tt} + G T^{tr} + H T^{t\theta} + I T^{t\phi}\right) \\ & + \left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\left(F T^{rt} + G T^{rr} + H T^{r\theta} + I T^{r\phi}\right) \\ & + \left(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}\right)\left(F T^{\theta t} + G T^{\theta r} + H T^{\theta\theta} + I T^{\theta\phi}\right) \\ & + \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\left(F T^{\phi t} + G T^{\phi r} + H T^{\phi\theta} + I T^{\phi\phi}\right) \end{aligned}$$

$$= F\Big[\frac{1}{c}\frac{\partial T^{tt}}{\partial t} + \Big(\frac{\partial}{\partial r}+\frac{2}{r}\Big)T^{rt} + \Big(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}\Big)T^{\theta t} + \Big(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\Big)T^{\phi t}\Big]$$

$$+ G\Big[\frac{1}{c}\frac{\partial T^{tr}}{\partial t} + \Big(\frac{\partial}{\partial r}+\frac{2}{r}\Big)T^{rr} + \Big(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}\Big)T^{\theta r} + \Big(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\Big)T^{\phi r}\Big]$$

$$+ H\Big[\frac{1}{c}\frac{\partial T^{t\theta}}{\partial t} + \Big(\frac{\partial}{\partial r}+\frac{2}{r}\Big)T^{r\theta} + \Big(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}\Big)T^{\theta\theta} + \Big(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\Big)T^{\phi\theta}\Big]$$

$$+ I\Big[\frac{1}{c}\frac{\partial T^{t\phi}}{\partial t} + \Big(\frac{\partial}{\partial r}+\frac{2}{r}\Big)T^{r\phi} + \Big(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}\Big)T^{\theta\phi} + \Big(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\Big)T^{\phi\phi}\Big]$$

$$+ T^{tt}\Big[\frac{1}{c}\frac{\partial F}{\partial t}\Big] + T^{tr}\Big[\frac{1}{c}\frac{\partial G}{\partial t}\Big] + T^{t\theta}\Big[\frac{1}{c}\frac{\partial H}{\partial t}\Big] + T^{t\phi}\Big[\frac{1}{c}\frac{\partial I}{\partial t}\Big]$$

$$+ T^{rt}\Big[\frac{\partial F}{\partial r}\Big] + T^{rr}\Big[\frac{\partial G}{\partial r}\Big] + T^{r\theta}\Big[\frac{\partial H}{\partial r}\Big] + T^{r\phi}\Big[\frac{\partial I}{\partial r}\Big]$$

$$+ T^{\theta t}\Big[\frac{1}{r}\frac{\partial F}{\partial\theta}\Big] + T^{\theta r}\Big[\frac{1}{r}\frac{\partial G}{\partial\theta}\Big] + T^{\theta\theta}\Big[\frac{1}{r}\frac{\partial H}{\partial\theta}\Big] + T^{\theta\phi}\Big[\frac{1}{r}\frac{\partial I}{\partial\theta}\Big]$$

$$+ T^{\phi t}\Big[\frac{1}{r\sin\theta}\frac{\partial F}{\partial\phi}\Big] + T^{\phi r}\Big[\frac{1}{r\sin\theta}\frac{\partial G}{\partial\phi}\Big] + T^{\phi\theta}\Big[\frac{1}{r\sin\theta}\frac{\partial H}{\partial\phi}\Big] + T^{\phi\phi}\Big[\frac{1}{r\sin\theta}\frac{\partial I}{\partial\phi}\Big]$$

Invoking equations (V.B.3) through (V.B.6), one obtains:

(V.B.9)

$$\text{Zero} = F\Big[\text{Zero}\Big]$$

$$+ G\Big[\frac{1}{r}T^{\theta\theta}+\frac{1}{r}T^{\phi\phi}\Big]$$

$$+ H\Big[-\frac{1}{r}T^{\theta r}+\frac{\cos\theta}{r\sin\theta}T^{\phi\phi}\Big]$$

$$+ I\Big[-\frac{1}{r}T^{\phi r}-\frac{\cos\theta}{r\sin\theta}T^{\phi\theta}\Big]$$

$$+ T^{tt}\Big[\frac{1}{c}\frac{\partial F}{\partial t}\Big] + T^{tr}\Big[\frac{1}{c}\frac{\partial G}{\partial t}\Big] + T^{t\theta}\Big[\frac{1}{c}\frac{\partial H}{\partial t}\Big] + T^{t\phi}\Big[\frac{1}{c}\frac{\partial I}{\partial t}\Big]$$

$$+ T^{rt}\Big[\frac{\partial F}{\partial r}\Big] + T^{rr}\Big[\frac{\partial G}{\partial r}\Big] + T^{r\theta}\Big[\frac{\partial H}{\partial r}\Big] + T^{r\phi}\Big[\frac{\partial I}{\partial r}\Big]$$

$$+ T^{\theta t}\Big[\frac{1}{r}\frac{\partial F}{\partial\theta}\Big] + T^{\theta r}\Big[\frac{1}{r}\frac{\partial G}{\partial\theta}\Big] + T^{\theta\theta}\Big[\frac{1}{r}\frac{\partial H}{\partial\theta}\Big] + T^{\theta\phi}\Big[\frac{1}{r}\frac{\partial I}{\partial\theta}\Big]$$

$$+ T^{\phi t}\Big[\frac{1}{r\sin\theta}\frac{\partial F}{\partial\phi}\Big] + T^{\phi r}\Big[\frac{1}{r\sin\theta}\frac{\partial G}{\partial\phi}\Big] + T^{\phi\theta}\Big[\frac{1}{r\sin\theta}\frac{\partial H}{\partial\phi}\Big] + T^{\phi\phi}\Big[\frac{1}{r\sin\theta}\frac{\partial I}{\partial\phi}\Big]$$

Collect terms to obtain:

$$\begin{aligned}\text{Zero} \;=\; & T^{tt}\Big[\frac{1}{c}\frac{\partial F}{\partial t}\Big] + T^{tr}\Big[\frac{1}{c}\frac{\partial G}{\partial t}\Big] + T^{t\theta}\Big[\frac{1}{c}\frac{\partial H}{\partial t}\Big] + T^{t\phi}\Big[\frac{1}{c}\frac{\partial I}{\partial t}\Big] \\ & + T^{rt}\Big[\frac{\partial F}{\partial r}\Big] + T^{rr}\Big[\frac{\partial G}{\partial r}\Big] + T^{r\theta}\Big[\frac{\partial H}{\partial r}\Big] + T^{r\phi}\Big[\frac{\partial I}{\partial r}\Big] \\ & + T^{\theta t}\Big[\frac{1}{r}\frac{\partial F}{\partial \theta}\Big] + T^{\theta r}\Big[\frac{1}{r}\frac{\partial G}{\partial \theta} - \frac{1}{r}H\Big] + T^{\theta\theta}\Big[\frac{1}{r}\frac{\partial H}{\partial \theta} + \frac{1}{r}G\Big] + T^{\theta\phi}\Big[\frac{1}{r}\frac{\partial I}{\partial \theta}\Big] \\ & + T^{\phi t}\Big[\frac{1}{r\sin\theta}\frac{\partial F}{\partial \phi}\Big] + T^{\phi r}\Big[\frac{1}{r\sin\theta}\frac{\partial G}{\partial \phi} - \frac{1}{r}I\Big] + T^{\phi\theta}\Big[\frac{1}{r\sin\theta}\frac{\partial H}{\partial \phi} - \frac{\cos\theta}{r\sin\theta}I\Big] \\ & + T^{\phi\phi}\Big[\frac{1}{r\sin\theta}\frac{\partial I}{\partial \phi} + \frac{1}{r}G + \frac{\cos\theta}{r\sin\theta}H\Big] \end{aligned} \qquad \text{(V.B.10)}$$

Since the above relation must hold identically, and since there is no *a priori* knowledge of linear relationships among the various $T^{\mu\nu}$ terms, it is required that each bracketted expression must itself equal zero. We thus are confronted with sixteen coupled partial differential equations whose solutions must be determined. Fortunately, the solution techniques are straightforward and need not be reproduced here. Suffice it to say that there are four linearly independent solutions which, for the sake of notational brevity, are listed in columnar format below:

$$\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 0 & \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ 0 & -\sin\phi & \cos\phi & 0 \end{pmatrix} \qquad \text{(V.B.11)}$$

However, if the $T^{\mu\nu}$ matrix displays any *a priori* symmetries, the requirement that all sixteen bracketted expressions must themselves be individually set to zero can be relaxed, thus opening the options for more solutions.

Two $T^{\mu\nu}$ symmetries of frequent appearance in electrodynamics are the following:

* Symmetric-Traceless† $T^{\mu\nu}$
* Antisymmetric $T^{\mu\nu}$

The premier example of the first type of symmetry is the electromagnetic energy-momentum-stress tensor in source-free space. Specifically, this is equation (V.A.9), or when written out in terms of its electric and magnetic field components, equation (IV.E.5) divided through by $1/4\pi$, with the R.H.S.'s of both equations set to zero.

The premier example of the second type of symmetry is the homogeneous pair of Maxwell equations, (V.A.5) .

Both these symmetry types will be examined in turn.

† Traceless in the Minkowskian sense, $T^{tt} - T^{rr} - T^{\theta\theta} - T^{\phi\phi} = 0$.

Examine first the case of symmetric-traceless $T^{\mu\nu}$.

In this case, we have equation (V.B.9) as always, but with the additional *a priori* knowledge that $T^{\theta\theta} = T^{tt} - T^{rr} - T^{\phi\phi}$ and that $T^{\mu\nu} = T^{\nu\mu}$. These relations can be used to eliminate the $T^{\theta\theta}$ term as well as the six $T^{\mu\nu}$ terms below the diagonal. After combining terms appropriately, equation (V.B.9) reduces to:

$$\begin{aligned}
\text{Zero} \;=\;& T^{tt}\Big[\frac{1}{c}\frac{\partial F}{\partial t}+\frac{1}{r}G+\frac{1}{r}\frac{\partial H}{\partial \theta}\Big] \\
&+\; T^{tr}\Big[\frac{\partial F}{\partial r}+\frac{1}{c}\frac{\partial G}{\partial t}\Big] \\
&+\; T^{t\theta}\Big[\frac{1}{r}\frac{\partial F}{\partial \theta}+\frac{1}{c}\frac{\partial H}{\partial t}\Big] \\
&+\; T^{t\phi}\Big[\frac{1}{r\sin\theta}\frac{\partial F}{\partial \phi}+\frac{1}{c}\frac{\partial I}{\partial t}\Big] \\
&+\; T^{rr}\Big[\frac{\partial G}{\partial r}-\frac{1}{r}G-\frac{1}{r}\frac{\partial H}{\partial \theta}\Big] \\
&+\; T^{r\theta}\Big[\frac{1}{r}\frac{\partial G}{\partial \theta}+\frac{\partial H}{\partial r}-\frac{1}{r}H\Big] \\
&+\; T^{r\phi}\Big[\frac{1}{r\sin\theta}\frac{\partial G}{\partial \phi}+\frac{\partial I}{\partial r}-\frac{1}{r}I\Big] \\
&+\; T^{\theta\phi}\Big[\frac{1}{r\sin\theta}\frac{\partial H}{\partial \phi}+\frac{1}{r}\frac{\partial I}{\partial \theta}-\frac{\cos\theta}{r\sin\theta}I\Big] \\
&+\; T^{\phi\phi}\Big[-\frac{1}{r}\frac{\partial H}{\partial \theta}+\frac{\cos\theta}{r\sin\theta}H+\frac{1}{r\sin\theta}\frac{\partial I}{\partial \phi}\Big]
\end{aligned} \qquad \text{(V.B.12)}$$

Since we require (V.B.12) to hold identically, all the bracketted expressions must individually be set to zero.

(V.B.13)

$$\frac{1}{c}\frac{\partial F}{\partial t}+\frac{1}{r}G+\frac{1}{r}\frac{\partial H}{\partial \theta} = 0 \quad \text{(a)}$$

$$\frac{\partial F}{\partial r}+\frac{1}{c}\frac{\partial G}{\partial t} = 0 \quad \text{(b)}$$

$$\frac{1}{r}\frac{\partial F}{\partial \theta}+\frac{1}{c}\frac{\partial H}{\partial t} = 0 \quad \text{(c)}$$

$$\frac{1}{r\sin\theta}\frac{\partial F}{\partial \phi}+\frac{1}{c}\frac{\partial I}{\partial t} = 0 \quad \text{(d)}$$

$$\frac{\partial G}{\partial r}-\frac{1}{r}G-\frac{1}{r}\frac{\partial H}{\partial \theta} = 0 \quad \text{(e)}$$

$$\frac{1}{r}\frac{\partial G}{\partial \theta}+\frac{\partial H}{\partial r}-\frac{1}{r}H = 0 \quad \text{(f)}$$

$$\frac{1}{r\sin\theta}\frac{\partial G}{\partial \phi}+\frac{\partial I}{\partial r}-\frac{1}{r}I = 0 \quad \text{(g)}$$

$$\frac{1}{r\sin\theta}\frac{\partial H}{\partial \phi}+\frac{1}{r}\frac{\partial I}{\partial \theta}-\frac{\cos\theta}{r\sin\theta}I = 0 \quad \text{(h)}$$

$$-\frac{1}{r}\frac{\partial H}{\partial \theta}+\frac{\cos\theta}{r\sin\theta}H+\frac{1}{r\sin\theta}\frac{\partial I}{\partial \phi} = 0 \quad \text{(i)}$$

Plug equation (e) into (a) and re-arrange terms to obtain:

(V.B.14)

$$\frac{1}{c}\frac{\partial F}{\partial t} = -\frac{\partial G}{\partial r} \tag{a}$$

$$\frac{\partial F}{\partial r} = -\frac{1}{c}\frac{\partial G}{\partial t} \tag{b}$$

$$\frac{1}{r}\frac{\partial F}{\partial \theta} = -\frac{1}{c}\frac{\partial H}{\partial t} \tag{c}$$

$$\frac{1}{r\sin\theta}\frac{\partial F}{\partial \phi} = -\frac{1}{c}\frac{\partial I}{\partial t} \tag{d}$$

$$\frac{\partial}{\partial r}\left(\frac{1}{r}G\right) = \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}H\right) \tag{e}$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}G\right) = -\frac{\partial}{\partial r}\left(\frac{1}{r}H\right) \tag{f}$$

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(\frac{1}{r}G\right) = -\frac{\partial}{\partial r}\left(\frac{1}{r}I\right) \tag{g}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}H\right) = -\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}I\right) \tag{h}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}\left(\frac{1}{\sin\theta}I\right) = \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}H\right) \tag{i}$$

Although these nine coupled partial differential equations are a bit more challenging to solve than the sixteen equations of the previous case, the solution techniques are much the same and need not be reproduced here. Sixteen linearly independent solutions (including one "trivial" solution) are obtained.

The sixteen solutions, including the trivial solution $(F,G,H,I) = (0,0,0,0)$, are intentionally grouped into four groups of four, the significance of which shall be made clear later on. In columnar format, these four groups of solutions are:

$$\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 0 & \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ 0 & -\sin\phi & \cos\phi & 0 \end{pmatrix} \tag{V.B.15}$$

$$\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -kr\sin\phi & kr\cos\phi & 0 \\ 0 & -kr\cos\theta\cos\phi & -kr\cos\theta\sin\phi & kr\sin\theta \end{pmatrix} \tag{V.B.16}$$

$$\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} = \begin{pmatrix} \omega t & kr\sin\theta\cos\phi & kr\sin\theta\sin\phi & kr\cos\theta \\ -kr & -\omega t\sin\theta\cos\phi & -\omega t\sin\theta\sin\phi & -\omega t\cos\theta \\ 0 & -\omega t\cos\theta\cos\phi & -\omega t\cos\theta\sin\phi & \omega t\sin\theta \\ 0 & \omega t\sin\phi & -\omega t\cos\phi & 0 \end{pmatrix} \tag{V.B.17}$$

$$\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} = \left(\begin{matrix} (k^2r^2+\omega^2t^2) & 2(kr)(\omega t)\sin\theta\cos\phi \\ -2(kr)(\omega t) & -(k^2r^2+\omega^2t^2)\sin\theta\cos\phi \\ 0 & (k^2r^2-\omega^2t^2)\cos\theta\cos\phi \\ 0 & -(k^2r^2-\omega^2t^2)\sin\phi \end{matrix} \right. \tag{V.B.18}$$

$$\left. \begin{matrix} 2(kr)(\omega t)\sin\theta\sin\phi & 2(kr)(\omega t)\cos\theta \\ -(k^2r^2+\omega^2t^2)\sin\theta\sin\phi & -(k^2r^2+\omega^2t^2)\cos\theta \\ (k^2r^2-\omega^2t^2)\cos\theta\sin\phi & -(k^2r^2-\omega^2t^2)\sin\theta \\ (k^2r^2-\omega^2t^2)\cos\phi & 0 \end{matrix} \right)$$

For any of these sixteen solutions (F,G,H,I), we have that:

$$\left(\frac{1}{c}\frac{\partial}{\partial t}\,,\,\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)\,,\,\left(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}\right)\,,\,\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\right)\cdot$$

$$\cdot\left[\begin{pmatrix} T^{tt} & T^{tr} & T^{t\theta} & T^{t\phi} \\ T^{rt} & T^{rr} & T^{r\theta} & T^{r\phi} \\ T^{\theta t} & T^{\theta r} & T^{\theta\theta} & T^{\theta\phi} \\ T^{\phi t} & T^{\phi r} & T^{\phi\theta} & T^{\phi\phi} \end{pmatrix}\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix}\right] = \text{Zero} \qquad \text{(V.B.19)}$$

We have thus extracted sixteen conserved quantities (including a trivial quantity "Zero") from the symmetric-traceless tensor $T^{\mu\nu}$.

In the case where $T^{\mu\nu}$ is specifically identified with the energy-momentum tensor, the conserved quantities assume actual physical significance, as will be discussed in the upcoming section.

But first, let us turn attention to the other case of physical interest, namely, antisymmetric $T^{\mu\nu}$.

The derivations for the antisymmetric $T^{\mu\nu}$ case proceed in exact analogy to the symmetric case. To avoid confusion between the two cases, the auxiliary functions (F,G,H,I) will be renamed (P,Q,R,S). Thus, for the antisymmetric case, in place of (V.B.7), we have:

$$\begin{aligned} P &= P(t,r,\theta,\phi) \\ Q &= Q(t,r,\theta,\phi) \\ R &= R(t,r,\theta,\phi) \\ S &= S(t,r,\theta,\phi) \end{aligned} \qquad \text{(V.B.20)}$$

such that:

$$\text{Zero} \;=\; \Big(\frac{1}{c}\frac{\partial}{\partial t}\,,\,\big(\frac{\partial}{\partial r}+\frac{2}{r}\big)\,,\,\big(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}\big)\,,\,\big(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\big)\Big)\,\cdot$$

$$\cdot\left[\begin{pmatrix} T^{tt} & T^{tr} & T^{t\theta} & T^{t\phi} \\ T^{rt} & T^{rr} & T^{r\theta} & T^{r\phi} \\ T^{\theta t} & T^{\theta r} & T^{\theta\theta} & T^{\theta\phi} \\ T^{\phi t} & T^{\phi r} & T^{\phi\theta} & T^{\phi\phi} \end{pmatrix}\begin{pmatrix} P \\ Q \\ R \\ S \end{pmatrix}\right] \qquad \text{(V.B.21)}$$

In this instance, however, it is known *a priori* that $T^{\mu\nu} = -T^{\nu\mu}$, implying also that diagonal terms $T^{\mu\mu}$ are identically zero.

We have equation (V.B.9) at our disposal just as before but with (F,G,H,I) replaced with (P,Q,R,S), respectively. This time the various bracketted terms combine linearly to give:

$$\begin{aligned} \text{Zero} \;=\;& T^{tr}\Big[\frac{1}{c}\frac{\partial Q}{\partial t}-\frac{\partial P}{\partial r}\Big] \qquad \text{(V.B.22)} \\ +\;& T^{t\theta}\Big[\frac{1}{c}\frac{\partial R}{\partial t}-\frac{1}{r}\frac{\partial P}{\partial\theta}\Big] \\ +\;& T^{t\phi}\Big[\frac{1}{c}\frac{\partial S}{\partial t}-\frac{1}{r\sin\theta}\frac{\partial P}{\partial\phi}\Big] \\ +\;& T^{r\theta}\Big[\frac{\partial R}{\partial r}-\frac{1}{r}\frac{\partial Q}{\partial\theta}+\frac{1}{r}R\Big] \\ +\;& T^{r\phi}\Big[\frac{\partial S}{\partial r}-\frac{1}{r\sin\theta}\frac{\partial Q}{\partial\phi}+\frac{1}{r}S\Big] \\ +\;& T^{\theta\phi}\Big[\frac{1}{r}\frac{\partial S}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial R}{\partial\phi}+\frac{\cos\theta}{r\sin\theta}S\Big] \end{aligned}$$

Once again, since we require identicality, each bracketted coefficient must equal zero.

(V.B.23)

$$\frac{1}{c}\frac{\partial Q}{\partial t} - \frac{\partial P}{\partial r} = 0 \quad \text{(a)}$$

$$\frac{1}{c}\frac{\partial R}{\partial t} - \frac{1}{r}\frac{\partial P}{\partial \theta} = 0 \quad \text{(b)}$$

$$\frac{1}{c}\frac{\partial S}{\partial t} - \frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi} = 0 \quad \text{(c)}$$

$$\frac{\partial R}{\partial r} - \frac{1}{r}\frac{\partial Q}{\partial \theta} + \frac{1}{r}R = 0 \quad \text{(d)}$$

$$\frac{\partial S}{\partial r} - \frac{1}{r\sin\theta}\frac{\partial Q}{\partial \phi} + \frac{1}{r}S = 0 \quad \text{(e)}$$

$$\frac{1}{r}\frac{\partial S}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial R}{\partial \phi} + \frac{\cos\theta}{r\sin\theta}S = 0 \quad \text{(f)}$$

After a little manipulation, one obtains:

(V.B.24)

$$\frac{1}{c}\frac{\partial Q}{\partial t} = \frac{\partial P}{\partial r} \quad \text{(a)}$$

$$\frac{1}{c}\frac{\partial}{\partial t}(rR) = \frac{\partial P}{\partial \theta} \quad \text{(b)}$$

$$\frac{1}{c}\frac{\partial}{\partial t}(r\sin\theta S) = \frac{\partial P}{\partial \phi} \quad \text{(c)}$$

$$\frac{\partial}{\partial r}(rR) = \frac{\partial Q}{\partial \theta} \quad \text{(d)}$$

$$\frac{\partial}{\partial r}(r\sin\theta S) = \frac{\partial Q}{\partial \phi} \quad \text{(e)}$$

$$\frac{\partial}{\partial \theta}(r\sin\theta S) = \frac{\partial}{\partial \phi}(rR) \quad \text{(f)}$$

The above system is underdetermined, meaning that there are not enough equations to specify unique solutions. So, instead of obtaining a finite number discrete solutions as in the previous cases, the above system admits an infinite family of solutions interrelated as indicated below:

$$
\begin{aligned}
P &= \frac{1}{c}\frac{\partial Z}{\partial t} \\
Q &= \frac{\partial Z}{\partial r} \\
R &= \frac{1}{r}\frac{\partial Z}{\partial \theta} \\
S &= \frac{1}{r\sin\theta}\frac{\partial Z}{\partial \phi}
\end{aligned}
\qquad \text{(V.B.25)}
$$

where $Z = Z(ct,r,\theta,\phi)$

Thus, the 4-vector (P,Q,R,S) is a 4-gradient of an arbitrary function $Z(ct,r,\theta,\phi)$.

One is now in a position to assess the above three solution types a bit more carefully. Recall that (V.B.10) is the necessary and sufficient condition to guarantee the conservation relation (V.B.8) for the case of $T^{\mu\nu}$ having no particular symmetry properties. There are sixteen requirements imposed upon the functions (F,G,H,I), namely that the sixteen bracketted coefficients of (V.B.10) be identically set equal to zero. There is a strong parallelism between these sixteen requirements and the formulas of Section (IV.E) of this report. First, identify the functions (F,G,H,I) with the generic covariant vector $(\Omega_t,\Omega_r,\Omega_\theta,\Omega_\phi)$. Then observe that the sixteen bracketted expressions of (V.B.10) are identical to the the sixteen elements of the "covariant derivative matrix" of (IV.E.7). Thus, the original objective of this section could have been restated as follows:

Given a doubly-contravariant tensor $T^{\mu\nu}$ that satisfies relation (V.B.1), the set of covariant vectors $(\Omega_t, \Omega_r, \Omega_\theta, \Omega_\phi)$ that would be necessary to guarantee the conservation relation (V.B.8) are those whose sixteen "covariant derivatives" are all equal to zero. These vectors are the curvilinear analogs of constant or "straight" vectors in Cartesian systems. The full solution set is given in (V.B.11).

Next, focus attention on the conditions necessary to guarantee the conservation relation (V.B.8) for *symmetric-traceless* $T^{\mu\nu}$. In this case, nine requirements are imposed on the functions (F, G, H, I), as given by the nine equations of (V.B.14). Upon making the identification $(F, G, H, I) = (\Omega_t, \Omega_r, \Omega_\theta, \Omega_\phi)$, it is found that these nine requirements are identically equivalent to setting the nine elements of the *symmetric-traceless* part of equation (IV.E.7) to zero. These nine elements are arrayed out explicitly in the third matrix of (IV.E.8).

Thus, stated in covariant language, one would say that given a *symmetric-traceless* doubly-contravariant tensor $T^{\mu\nu}$ that satisfies relation (V.B.1), the set of covariant vectors $(\Omega_t, \Omega_r, \Omega_\theta, \Omega_\phi)$ that would be necessary to guarantee the conservation relation (V.B.8) are those whose nine *symmetrized, traceless* "covariant derivatives" are all equal to zero. The full solution set is given in (V.B.15 thru 18).

Lastly, attention is turned to the case of *anti-symmetric* $T^{\mu\nu}$. In this case, six requirements are imposed on the functions (P,Q,R,S), as given by the six equations of (V.B.23). Upon making the identification $(P,Q,R,S) = (\Omega_t, \Omega_r, \Omega_\theta, \Omega_\phi)$, it is found that these six requirements are identically equivalent to setting the six elements of the *anti-symmetric* part of equation (IV.E.7) to zero. These six elements are arrayed out explicitly in the second matrix of (IV.E.8) and presented again in equations (IV.E.10) and (IV.E.11).

Stated in covariant language, one would say that given an *anti-symmetric* doubly-contravariant tensor $T^{\mu\nu}$ that satisfies the relation (V.B.1), the set of covariant vectors $(\Omega_t, \Omega_r, \Omega_\theta, \Omega_\phi)$ that would be necessary to guarantee the conservation relation (V.B.8) are those whose six *anti-symmetrized* "covariant derivatives" are equal to zero. The full solution set is embodied by the requirement that these vectors be 4-gradients. Refer specifically to (V.B.25).

Although not explicity examined in the previous parts of this section, it should not be too large a leap to consider the case of scalar-diagonal $T^{\mu\nu}$. The covariant vectors that would be necessary to guarantee the conservation relation (V.B.8) in this case would be those whose *diagonalized* component of "covariant derivatives" is set equal to zero. Specifically, this is equivalent to setting the first matrix of (IV.E.8) to zero. In this case, the required $(\Omega_t, \Omega_r, \Omega_\theta, \Omega_\phi)$ would have to be expressible as a 4-curl. Maxwell's equations (V.A.3) provide good examples of such vectors.

A note about continuity relations.

It has been previously mentioned that any relation of the form

$$-\frac{1}{c}\frac{\partial a^0}{\partial t} = \left(\frac{\partial}{\partial r}+\frac{2}{r}\right)a^1 + \left(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}\right)a^2 + \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)a^3$$

implies a conservation law for the quantity $\int a^0 dV$, or, alternatively, that a *continuity relation* exists for a^0. Both these expressions are explained below.

Integrate the above continuity equation over a volume V:

$$-\frac{1}{c}\frac{\partial}{\partial t}\int_V a^0\, dV = \int_V \left[\left(\frac{\partial}{\partial r}+\frac{2}{r}\right)a^1 + \left(\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{\cos\theta}{r\sin\theta}\right)a^2 + \left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)a^3\right] dV$$

(V.B.26)

Use Gauss's Divergence Theorem on the R.H.S. to obtain:

$$-\frac{1}{c}\frac{\partial A^0}{\partial t} = \oint_S \left(\hat{\mathrm{r}}a^1 + \hat{\theta}a^2 + \hat{\phi}a^3\right)\cdot\hat{\mathrm{n}}\, dS \qquad \text{(V.B.27)}$$

where

$A^0 = \int a^0 dV$

S = closed surface that bounds volume V

$\hat{\mathrm{n}}$ = *outward*-directed unit-normal to surface S

The above is the mathematical statement that the rate at which a quantity A^0 decreases within a volume V (L.H.S. of equation) equals the total amount expelled from that volume through bounding surface S (R.H.S. of equation).

Overall, no A^0 is lost or gained; hence, *conservation* of A^0. Similarly, there is no "tele-transport" of quantity A^0 from volume V_1 to volume V_2 without passing through intervening surface S; hence, *continuity* of A^0.

The 4-vector formalism is such that for any (a^0, a^1, a^2, a^3) satisfying the continuity equation (V.B.26), it is always the lead component a^0 that is the conserved density. The other three components are flux densities that account for passage through the bounding surface S.

Since attention usually focusses on the conserved physical quantities, it is typical to emphasize the lead component a^0 at the expense of the other three. Therefore, the quantities of interest in equations (V.B.19) are:

$$\begin{pmatrix} T^{tt} & T^{tr} & T^{t\theta} & T^{t\phi} \end{pmatrix} \begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} \tag{V.B.28}$$

This practice of considering only the first row of the core $T^{\mu\nu}$ matrix is utilized frequently in the upcoming section, where emphasis is placed on the conserved quantities themselves, and not the attendant flux terms.

C.) Conserved Electromagnetic Quantities

Although divergenceless 4-vectors have been established for various tensor types, those associated with the symmetric-traceless energy-momentum tensor (V.A.9) are the most compelling physically. The sixteen 4-vectors associated with this tensor, as catalogued in equations (V.B.15) through (V.B.18), have familiar physical interpretations.

These interpretations become clear when a transformation to the Cartesian coordinate system is performed. The transformation law, (IV.D.6), re-cast in a form more suitable to our purposes is:

$$\begin{pmatrix} u & cg_r & cg_\theta & cg_\phi \\ cg_r & \sigma_{rr} & \sigma_{r\theta} & \sigma_{r\phi} \\ cg_\theta & \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta\phi} \\ cg_\phi & \sigma_{\phi r} & \sigma_{\phi\theta} & \sigma_{\phi\phi} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 0 & \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ 0 & -\sin\phi & \cos\phi & 0 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} u & cg_x & cg_y & cg_z \\ cg_x & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ cg_y & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ cg_z & \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ 0 & \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ 0 & \cos\theta & -\sin\theta & 0 \end{pmatrix}$$

(V.C.1)

Consider the first set of divergenceless quantities, namely, those to be constructed from the $T^{\mu\nu}$ tensor and the four column vectors of (V.B.15), and transform them as prescribed to obtain:

$$\begin{pmatrix} u & cg_r & cg_\theta & cg_\phi \\ cg_r & \sigma_{rr} & \sigma_{r\theta} & \sigma_{r\phi} \\ cg_\theta & \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta\phi} \\ cg_\phi & \sigma_{\phi r} & \sigma_{\phi\theta} & \sigma_{\phi\phi} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 0 & \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ 0 & -\sin\phi & \cos\phi & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 0 & \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ 0 & -\sin\phi & \cos\phi & 0 \end{pmatrix} \cdot \qquad \text{(V.C.2)}$$

$$\cdot \begin{pmatrix} u & cg_x & cg_y & cg_z \\ cg_x & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ cg_y & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ cg_z & \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ 0 & \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ 0 & \cos\theta & -\sin\theta & 0 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 0 & \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ 0 & -\sin\phi & \cos\phi & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 0 & \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ 0 & -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} u & cg_x & cg_y & cg_z \\ cg_x & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ cg_y & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ cg_z & \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In the spirit of equation (V.B.28), restrict attention to only the first row of the above matrix equation. This is adequate since it is only the conserved quantities that interest us here, and not the attendant flux terms.

One obtains for the first set of conserved quantities:

$$\begin{pmatrix} u & cg_r & cg_\theta & cg_\phi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 0 & \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ 0 & -\sin\phi & \cos\phi & 0 \end{pmatrix} = \tag{V.C.3}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u & cg_x & cg_y & cg_z \\ cg_x & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ cg_y & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ cg_z & \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} = \begin{pmatrix} u & cg_x & cg_y & cg_z \end{pmatrix}$$

From the discussion that follows equation (V.B.27), it is evident that the first set of conserved quantities, that is, those constructed from $T^{\mu\nu}$ and the four column vectors of (V.B.15), are going to be:

$$U = \int_V u\,dV \tag{V.C.4}$$

$$cG_x = \int_V cg_x\,dV \tag{V.C.5}$$

$$cG_y = \int_V cg_y\,dV \tag{V.C.6}$$

$$cG_z = \int_V cg_z\,dV \tag{V.C.7}$$

The above quantities represent electromagnetic energy and the three Cartesian components of electromagnetic momentum (multiplied by the speed of light, c).

Proceeding in exactly the same manner for the second set of divergenceless quantities, that is, those constructed from $T^{\mu\nu}$ and the four column vectors of (V.B.16), one obtains:

$$\begin{pmatrix} u & cg_r & cg_\theta & cg_\phi \\ cg_r & \sigma_{rr} & \sigma_{r\theta} & \sigma_{r\phi} \\ cg_\theta & \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta\phi} \\ cg_\phi & \sigma_{\phi r} & \sigma_{\phi\theta} & \sigma_{\phi\phi} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -kr\sin\phi & kr\cos\phi & 0 \\ 0 & -kr\cos\theta\cos\phi & -kr\cos\theta\sin\phi & kr\sin\theta \end{pmatrix} = \cdot$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 0 & \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ 0 & -\sin\phi & \cos\phi & 0 \end{pmatrix} \cdot \tag{V.C.8}$$

$$\cdot \begin{pmatrix} u & cg_x & cg_y & cg_z \\ cg_x & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ cg_y & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ cg_z & \sigma_{zx} & \sigma zy & \sigma_{zz} \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ 0 & \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ 0 & \cos\theta & -\sin\theta & 0 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -kr\sin\phi & kr\cos\phi & 0 \\ 0 & -kr\cos\theta\cos\phi & -kr\cos\theta\sin\phi & kr\sin\theta \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 0 & \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ 0 & -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} u & cg_x & cg_y & cg_z \\ cg_x & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ cg_y & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ cg_z & \sigma_{zx} & \sigma zy & \sigma_{zz} \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & kr\cos\theta & -kr\sin\theta\sin\phi \\ 0 & -kr\cos\theta & 0 & kr\sin\theta\cos\phi \\ 0 & kr\sin\theta\sin\phi & -kr\sin\theta\cos\phi & 0 \end{pmatrix} \qquad \text{(V.C.8)}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 0 & \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ 0 & -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} u & cg_x & cg_y & cg_z \\ cg_x & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ cg_y & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ cg_z & \sigma_{zx} & \sigma zy & \sigma_{zz} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & kz & -ky \\ 0 & -kz & 0 & kx \\ 0 & ky & -kx & 0 \end{pmatrix}$$

Once again, we need only concern ourselves with the top row of the above matrix equation. We obtain our second set of conserved quantities:

$$\begin{pmatrix} u & cg_r & cg_\theta & cg_\phi \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -kr\sin\phi & kr\cos\phi & 0 \\ 0 & -kr\cos\theta\cos\phi & -kr\cos\theta\sin\phi & kr\sin\theta \end{pmatrix} = \qquad \text{(V.C.9)}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u & cg_x & cg_y & cg_z \\ cg_x & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ cg_y & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ cg_z & \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & kz & -ky \\ 0 & -kz & 0 & kx \\ 0 & ky & -kx & 0 \end{pmatrix}$$

$$= \begin{pmatrix} u & cg_x & cg_y & cg_z \\ & & & \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & kz & -ky \\ 0 & -kz & 0 & kx \\ 0 & ky & -kx & 0 \end{pmatrix} \tag{V.C.9}$$

$$= \omega(0, m_x, m_y, m_z)$$

Therefore it follows that the second set of conserved quantities, that is, those constructed from $T^{\mu\nu}$ and the four column vectors of (V.B.16), are going to be:

$$0 = \int_V 0\, dV \tag{V.C.10}$$

$$\omega M_x = \int_V \omega m_x\, dV \tag{V.C.11}$$

$$\omega M_y = \int_V \omega m_y\, dV \tag{V.C.12}$$

$$\omega M_z = \int_V \omega m_z\, dV \tag{V.C.13}$$

The above represent the "trivial" conserved quantity (Zero) and the three Cartesian components of angular momentum (multiplied by angular frequency ω).

It is interesting to note that this is the second time that mathematical developments have yielded conservation laws for *Cartesian* components of select vector quantities. Cartesian components are somehow singled out in Maxwellian electrodynamics, even when working in "impartial" non-Cartesian systems such as the one used here. This trend, in fact, holds globally, as will be demonstrated in the remaining portions of this section.

Now that the general scheme of calculation has been outlined, it should not be necessary to have to repeat the calculations for the remaining two expressions (V.B.17) and (V.B.18), but rather merely quote the results. The derivations proceed exactly as they did in the previous two cases.

For the four divergenceless expressions of (V.B.17), one obtains the following four conserved quantities:

$$\omega N_t = \omega \int_V \left(tu - \vec{r} \cdot \vec{g} \right) dV \tag{V.C.14}$$

$$\omega N_x = \omega \int_V \left(x\left(\frac{u}{c}\right) - ct\left(g_x\right)\right) dV \tag{V.C.15}$$

$$\omega N_y = \omega \int_V \left(y\left(\frac{u}{c}\right) - ct\left(g_y\right)\right) dV \tag{V.C.16}$$

$$\omega N_z = \omega \int_V \left(z\left(\frac{u}{c}\right) - ct\left(g_z\right)\right) dV \tag{V.C.17}$$

The above four quantities have no mechanical analog such as "Energy" or "Angular Momentum" from which to borrow terminology and notation. However, they possess somewhat the same mathematical form as mechanical Action:

$$I = \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} (p_i \dot{q}_i - H)\,dt$$

The nomenclature (N_t, N_x, N_y, N_z) is purely arbitrary.

For the divergenceless quantities constructed from $T^{\mu\nu}$ and the four column vectors of (V.B.18), one obtains:

$$k\omega\int_V\Big((r^2+c^2t^2)\Big(\frac{u}{c}\Big) - 2ct(\vec{r}\cdot\vec{g})\Big)dV \qquad \text{(V.C.18)}$$

$$k\omega\int_V\Big(2x(tu-\vec{r}\cdot\vec{g}) + (r^2-c^2t^2)g_x\Big)dV \qquad \text{(V.C.19)}$$

$$k\omega\int_V\Big(2y(tu-\vec{r}\cdot\vec{g}) + (r^2-c^2t^2)g_y\Big)dV \qquad \text{(V.C.20)}$$

$$k\omega\int_V\Big(2z(tu-\vec{r}\cdot\vec{g}) + (r^2-c^2t^2)g_z\Big)dV \qquad \text{(V.C.21)}$$

These last four conserved quantities are purely electromagnetic in origin, having no analog in mechanical systems.

D.) Sixteen Conservation Laws

One can now proceed with the explicit conservation laws as specified by (V.B.26). It will no longer be adequate to "retain the first row only" when constructing these laws; rather, all four terms of the relevant continuity relation must be utilized. Each divergenceless quantity will be examined in turn; no more combining them into related sets of four.

The general form of the conservation law is:

$$\Box \cdot \left[\begin{pmatrix} u & cg_r & cg_\theta & cg_\phi \\ cg_r & \sigma_{rr} & \sigma_{r\theta} & \sigma_{r\phi} \\ cg_\theta & \sigma_{\theta r} & \sigma_{\theta\theta} & \sigma_{\theta\phi} \\ cg_\phi & \sigma_{\phi r} & \sigma_{\phi\theta} & \sigma_{\phi\phi} \end{pmatrix} \begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} \right] = \text{Zero} \qquad \text{(V.D.1)}$$

Written out explicitly and integrated over a volume V, one has that:

$$-\frac{1}{c}\frac{\partial}{\partial t}\int_V \left[uF + cg_r G + cg_\theta H + cg_\phi I \right] dV = \qquad \text{(V.D.2)}$$

$$= \int_V \Big[\Big(\frac{\partial}{\partial r} + \frac{2}{r} \Big) \big(cg_r F + \sigma_{rr} G + \sigma_{r\theta} H + \sigma_{r\phi} I \big)$$

$$+ \Big(\frac{1}{r}\frac{\partial}{\partial \theta} + \frac{\cos\theta}{r\sin\theta} \Big) \big(cg_\theta F + \sigma_{\theta r} G + \sigma_{\theta\theta} H + \sigma_{\theta\phi} I \big)$$

$$+ \Big(\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \Big) \big(cg_\phi F + \sigma_{\phi r} G + \sigma_{\phi\theta} H + \sigma_{\phi\phi} I \big) \Big] dV$$

Applying Gauss's Divergence Theorem on the R.H.S. yields:

$$-\frac{1}{c}\frac{\partial}{\partial t}\int_V \left[uF + cg_r G + cg_\theta H + cg_\phi I\right] dV = \tag{V.D.3}$$

$$= \oint_S \Big[\hat{\mathrm{r}}\left(cg_r F + \sigma_{rr} G + \sigma_{r\theta} H + \sigma_{r\phi} I\right)$$

$$+ \hat{\theta}\left(cg_\theta F + \sigma_{\theta r} G + \sigma_{\theta\theta} H + \sigma_{\theta\phi} I\right)$$

$$+ \hat{\phi}\left(cg_\phi F + \sigma_{\phi r} G + \sigma_{\phi\theta} H + \sigma_{\phi\phi} I\right)\Big]\cdot \hat{\mathrm{n}}\, dS$$

The objective now is to insert allowed (F,G,H,I) solutions into the above relation to obtain spherical-coordinate representations of conservation laws. The physical quantity being conserved for each law has been tagged and identified in the previous section. Recall that these quantities are electromagnetic energy, momentum, angular-momentum, and so forth. It will prove informative to examine each conserved quantity in turn.

For the case that: $$\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{V.D.4}$$

one has, after reviewing (V.C.3) and (V.C.4), a conservation law for electromagnetic energy, U, times $1/c$. The time derivative on the L.H.S. of (V.D.3) for this choice of (F,G,H,I) is the rate (Watt) at which electromagnetic energy is decreasing within volume V, multiplied by $1/c$. The surface integral on the R.H.S. of (V.D.3) is the rate at which electromagnetic energy is ejected through bounding surface S multiplied by $1/c$. c is the speed of light in vacuum.

For the case that: $$\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ \sin\theta\cos\phi \\ \cos\theta\cos\phi \\ -\sin\phi \end{pmatrix} \tag{V.D.5}$$

one has, after reviewing (V.C.3) and (V.C.5), a conservation law for x-component of electromagnetic momentum, G_x. The time derivative on the L.H.S. of (V.D.3) for this choice of (F,G,H,I) is the rate (Newton) at which x-component of electromagnetic momentum is decreasing within volume V. The surface integral on the R.H.S. of (V.D.3) is the rate at which x-component of electromagnetic momentum is ejected through bounding surface S.

For the case that:
$$\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ \sin\theta\sin\phi \\ \cos\theta\sin\phi \\ \cos\phi \end{pmatrix} \tag{V.D.6}$$

one has, after reviewing (V.C.3) and (V.C.6), a conservation law for y-component of electromagnetic momentum, G_y. The time derivative on the L.H.S. of (V.D.3) for this choice of (F,G,H,I) is the rate (Newton) at which y-component of electromagnetic momentum is decreasing within volume V. The surface integral on the R.H.S. of (V.D.3) is the rate at which y-component of electromagnetic momentum is ejected through bounding surface S.

For the case that:
$$\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ \cos\theta \\ -\sin\theta \\ 0 \end{pmatrix} \tag{V.D.7}$$

one has, after reviewing (V.C.3) and (V.C.7), a conservation law for z-component of electromagnetic momentum, G_z. The time derivative on the L.H.S. of (V.D.3) for this choice of (F,G,H,I) is the rate (Newton) at which z-component of electromagnetic momentum is decreasing within volume V. The surface integral on the R.H.S. of (V.D.3) is the rate at which z-component of electromagnetic momentum is ejected through bounding surface S.

For the case that:

$$\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \text{(V.D.8)}$$

one has the "trivial" identity. Although seemingly not of mathematical interest, this solution does have its place in the scheme of things. But this issue shall not be pursued here; it will be deferred to a subsequent report.

For the case that:

$$\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -kr\sin\phi \\ -kr\cos\theta\cos\phi \end{pmatrix} \qquad \text{(V.D.9)}$$

one has, after reviewing (V.C.9) and (V.C.11), a conservation law for x-component of electromagnetic angular momentum, M_x, multiplied by wave number $k = \omega/c$. The time derivative on the L.H.S. of (V.D.3) for this choice of (F,G,H,I) is the rate at which x-component of electromagnetic angular momentum is decreasing within volume V, multiplied by wave number k. The surface integral on the R.H.S. of (V.D.3) is the rate at which x-component of electromagnetic angular momentum is ejected through bounding surface S, multiplied by wave number k.

For the case that:
$$\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ kr\cos\phi \\ -kr\cos\theta\sin\phi \end{pmatrix} \qquad \text{(V.D.10)}$$

one has, after reviewing (V.C.9) and (V.C.12), a conservation law for y-component of electromagnetic angular momentum, M_y, multiplied by wave number $k = \omega/c$. The time derivative on the L.H.S. of (V.D.3) for this choice of (F,G,H,I) is the rate at which y-component of electromagnetic angular momentum is decreasing within volume V, multipled by wave number k. The surface integral on the R.H.S. of (V.D.3)is the rate at which y-component of electromagnetic angular momentum is ejected through bounding surface S, multipled by wave number k.

For the case that:
$$\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ kr\sin\theta \end{pmatrix} \qquad \text{(V.D.11)}$$

one has, after reviewing (V.C.9) and (V.C.13), a conservation law for z-component of electromagnetic angular momentum, M_z, multiplied by wave number $k = \omega/c$. The time derivative on the L.H.S. of (V.D.3) for this choice of (F,G,H,I) is the rate at which z-component of electromagnetic angular momentum is decreasing within volume V, multipled by wave number k. The surface integral on the R.H.S. of (V.D.3) is the rate at which z-component of electromagnetic angular momentum is ejected through bounding surface S, multipled by wave number k.

Similarly:

For the cases that:
$$\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} = \begin{pmatrix} \omega t \\ -kr \\ 0 \\ 0 \end{pmatrix} \tag{V.D.12}$$

$$\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} = \begin{pmatrix} kr\sin\theta\cos\phi \\ -\omega t\sin\theta\cos\phi \\ -\omega t\cos\theta\cos\phi \\ \omega t\sin\phi \end{pmatrix} \tag{V.D.13}$$

$$\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} = \begin{pmatrix} kr\sin\theta\sin\phi \\ -\omega t\sin\theta\sin\phi \\ -\omega t\cos\theta\sin\phi \\ -\omega t\cos\phi \end{pmatrix} \tag{V.D.14}$$

$$\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} = \begin{pmatrix} kr\cos\theta \\ -\omega t\cos\theta \\ \omega t\sin\theta \\ 0 \end{pmatrix} \tag{V.D.15}$$

one has conservation laws for the electromagnetic quantities that were denoted N_t, N_x, N_y, N_z in equations (V.C.14 thru 17), multiplied by wave number k.

And similarly for the last four allowed vectors:

$$\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} = \begin{pmatrix} (k^2r^2+\omega^2t^2) \\ -2(kr)(\omega t) \\ 0 \\ 0 \end{pmatrix} \tag{V.D.16}$$

$$\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} = \begin{pmatrix} 2(kr)(\omega t)\sin\theta\cos\phi \\ -(k^2r^2+\omega^2t^2)\sin\theta\cos\phi \\ (k^2r^2-\omega^2t^2)\cos\theta\cos\phi \\ -(k^2r^2-\omega^2t^2)\sin\phi \end{pmatrix} \tag{V.D.17}$$

$$\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} = \begin{pmatrix} 2(kr)(\omega t)\sin\theta\sin\phi \\ -(k^2r^2+\omega^2t^2)\sin\theta\sin\phi \\ (k^2r^2-\omega^2t^2)\cos\theta\sin\phi \\ (k^2r^2-\omega^2t^2)\cos\phi \end{pmatrix} \tag{V.D.18}$$

$$\begin{pmatrix} F \\ G \\ H \\ I \end{pmatrix} = \begin{pmatrix} 2(kr)(\omega t)\cos\theta \\ -(k^2r^2+\omega^2t^2)\cos\theta \\ -(k^2r^2-\omega^2t^2)\sin\theta \\ 0 \end{pmatrix} \tag{V.D.19}$$

one has conservation laws for the electromagnetic quantities given in equations (V.C.18 thru 21).

Thus, fifteen non-trivial conservation laws have been extracted from the electromagnetic energy-momentum-stress tensor $T^{\mu\nu}$.

CHAPTER VI

DISCRETE SUM EXPRESSIONS FOR CONSERVED ELECTROMAGNETIC QUANTITIES

Fifteen conservation laws for various electrodynamic quantities have been presented in the previous chapter, of which seven will be investigated in this chapter. Recall that the tactic behind these laws is to isolate the time derivative of the conserved quantity on the L.H.S. of a continuity equation and to supply an appropriate surface integral on the R.H.S. Equation (V.D.3) is the applicable template for laws expressed in this in this form. Various functional choices for the column vector (F, G, H, I) as given in equations (V.D.4 thru 19) infer different conservation laws. It is incumbent upon us to convert these generic conservation laws into explicit expressions involving known aspects of the radiating system. This is achieved by defining a suitable boundary surface S for equation (V.D.3), replacing all u, $\vec{g}$, and $\vec{\vec{\sigma}}$ terms in the (V.D.3) surface integrand with their (V.A.6 thru 8) expansions in terms of $\vec{E}$ and $\vec{B}$, and then replacing all $\vec{E}$ and $\vec{B}$ terms with their Maxwellian solutions (III.F.11 thru 16) so as to perform the relevant surface integrations.

It is at this point that the recursion and orthogonality relations of Appendix Sections B, C, and D become indispensible.

It is my intent to reproduce calculations for the first conserved quantity only, *i.e.,* the quantity inferred by (V.D.3)/(V.D.4). In the interest of space, the calculations for the remaining six quantities will be side-stepped so that final results can be presented with no undue delay. The calculations asociated with the first conserved quantity should suffice to demonstrate the methodology for the other conserved quantities.

For the general situation of finitely-sized radiating objects, the relevant boundary surface is a pair of concentric spheres† of inner and outer radius R_1 and R_2.

Recall the general form for the seven conservation laws:

$$-\frac{1}{c}\frac{\partial A^0}{\partial t} = \oint_S \left(\hat{\mathrm{r}}\, a^1 + \hat{\theta}\, a^2 + \hat{\phi}\, a^3\right)\cdot \hat{\mathrm{n}}\, dS \tag{VI.1}$$

For the particular geometry selected, the element of differential surface area becomes:

$$dS = r^2 \sin\theta\, d\theta\, d\phi \tag{VI.2}$$

and the outwardly-directed unit-normals become:

$$\hat{\mathrm{n}} = +\hat{\mathrm{r}} \tag{VI.3}$$

on the outer $r = R_2$ sphere, and

$$\hat{\mathrm{n}} = -\hat{\mathrm{r}} \tag{VI.4}$$

on the inner $r = R_1$ sphere.

† The particular merits of this choice of S will be made clear in the discussion that concludes this section.

For this geometry, the conservation law of (V.D.3) reduces to:

$$-\frac{1}{c}\frac{\partial A^0}{\partial t} = \left[\int_0^{2\pi}\int_0^{\pi} a^1 r^2 \sin\theta \, d\theta \, d\phi\right]_{R_1}^{R_2} \qquad \text{(VI.5)}$$

That is to say, the surface integration is performed over the full range of θ and ϕ on the two surfaces $r = R_2$ and $r = R_1$. These results are then subtracted because $\hat{n} \cdot \hat{r} = +1$ on the $r = R_2$ surface and $\hat{n} \cdot \hat{r} = -1$ on the $r = R_1$ surface.

The corresponding flux quantity, denoted W_{A^0}, is defined as:

$$\frac{1}{c} W_{A^0} = \int_0^{2\pi}\int_0^{\pi} a^1 r^2 \sin\theta \, d\theta \, d\phi \qquad \text{(VI.6)}$$

It is these flux quantities that will be derived in the developments that follow.

The actual $-\partial A^0/\partial t$ value can be evaluated from the W_{A^0} quantity very simply by noting that:

$$-\frac{\partial A^0}{\partial t} = \left[W_{A^0}\right]_{R_1}^{R_2} \qquad \text{(VI.7)}$$

For electromagnetic energy, we have (I.32)/(I.33):

$$-\frac{1}{c}\frac{\partial U}{\partial t} = \oint_{S_1}^{S_2}\left[\hat{\mathrm{r}}\left(cg_r\right) + \hat{\theta}\left(cg_\theta\right) + \hat{\phi}\left(cg_\phi\right)\right]\cdot\hat{\mathrm{r}}\, r^2 \sin\theta \, d\theta d\phi$$

$$= \left[\int_0^{2\pi}\int_0^{\pi} cg_r\, r^2 \sin\theta\, d\theta d\phi\right]_{R_1}^{R_2} \tag{VI.8}$$

The corresponding energy flux W_U is given as:

$$\frac{1}{c}W_U = \int_0^{2\pi}\int_0^{\pi} cg_r\, r^2 \sin\theta\, d\theta d\phi \tag{VI.9}$$

Use equation (I.19) to express cg_r in terms of E_i, B_j:

$$\frac{1}{c}W_U = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(E_\theta B_\phi - E_\phi B_\theta\right) r^2 \sin\theta\, d\theta d\phi$$

Use equations (III.F.12, 13, 15, 16) to express E_i and B_j as functions of (r,θ,ϕ):

$$\frac{1}{c}W_U = \frac{1}{16\pi}\int_0^{2\pi}\int_0^{\pi} \cdot \tag{VI.9$'$}$$

$$\cdot \Bigg[\Bigg(\sum_{l=0}^{\infty}\sum_{m=-l}^{l} g_{lm}\Big(\frac{dh_l^{(1)}}{dr}+\frac{h_l^{(1)}}{r}\Big)\frac{dP_l^m}{d\theta}e^{im\phi}e^{-i\omega t} - kd_{lm}h_l^{(1)}\frac{m}{\sin\theta}P_l^m e^{im\phi}e^{-i\omega t}$$

$$+ g_{lm}^*\Big(\frac{dh_l^{(2)}}{dr}+\frac{h_l^{(2)}}{r}\Big)\frac{dP_l^m}{d\theta}e^{-im\phi}e^{i\omega t} - kd_{lm}^*h_l^{(2)}\frac{m}{\sin\theta}P_l^m e^{-im\phi}e^{i\omega t}\Bigg)\cdot$$

$$\cdot\Bigg(\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} id_{l'm'}\Big(\frac{dh_{l'}^{(1)}}{dr}+\frac{h_{l'}^{(1)}}{r}\Big)\frac{m'}{\sin\theta}P_{l'}^{m'}e^{im'\phi}e^{-i\omega t} + ikg_{l'm'}h_{l'}^{(1)}\frac{dP_{l'}^{m'}}{d\theta}e^{im'\phi}e^{-i\omega t}$$

$$-id_{l'm'}^*\Big(\frac{dh_{l'}^{(2)}}{dr}+\frac{h_{l'}^{(2)}}{r}\Big)\frac{m'}{\sin\theta}P_{l'}^{m'}e^{-im'\phi}e^{i\omega t} - ikg_{l'm'}^*h_{l'}^{(2)}\frac{dP_{l'}^{m'}}{d\theta}e^{-im'\phi}e^{i\omega t}\Bigg)$$

$$-\Bigg(\sum_{l=0}^{\infty}\sum_{m=-l}^{l} ig_{lm}\Big(\frac{dh_l^{(1)}}{dr}+\frac{h_l^{(1)}}{r}\Big)\frac{m}{\sin\theta}P_l^m e^{im\phi}e^{-i\omega t} - ikd_{lm}h_l^{(1)}\frac{dP_l^m}{d\theta}e^{im\phi}e^{-i\omega t}$$

$$-ig_{lm}^*\Big(\frac{dh_l^{(2)}}{dr}+\frac{h_l^{(2)}}{r}\Big)\frac{m}{\sin\theta}P_l^m e^{-im\phi}e^{i\omega t} + ikd_{lm}^*h_l^{(1)}\frac{dP_l^m}{d\theta}e^{-im\phi}e^{i\omega t}\Bigg)\cdot$$

$$\cdot\Bigg(\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l'} d_{l'm'}\Big(\frac{dh_{l'}^{(1)}}{dr}+\frac{h_{l'}^{(1)}}{r}\Big)\frac{dP_{l'}^{m'}}{d\theta}e^{im'\phi}e^{-i\omega t} + kg_{l'm'}h_{l'}^{(1)}\frac{m'}{\sin\theta}P_{l'}^{m'}e^{im'\phi}e^{-i\omega t}$$

$$+d_{l'm'}^*\Big(\frac{dh_{l'}^{(2)}}{dr}+\frac{h_{l'}^{(2)}}{r}\Big)\frac{dP_{l'}^{m'}}{d\theta}e^{-im'\phi}e^{i\omega t} + kg_{l'm'}^*h_{l'}^{(2)}\frac{m'}{\sin\theta}P_{l'}^{m'}e^{-im'\phi}e^{i\omega t}\Bigg)\Bigg]\cdot$$

$$\cdot\, r^2\sin\theta\, d\theta\, d\phi$$

Combine terms and use equation (App.D1.1) to obtain:

$$\frac{1}{c} W_U = \frac{1}{16\pi} \int_0^{\pi} \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} \sum_{m=-l}^{l} \sum_{m'=-l'}^{l'} r^2 \sin\theta \, d\theta \cdot \qquad \text{(VI.9')}$$

$$\cdot \Bigg(i g_{lm} d_{l'm'} \left(\frac{dh_l^{(1)}}{dr} + \frac{h_l^{(1)}}{r} \right) \left(\frac{dh_{l'}^{(1)}}{dr} + \frac{h_{l'}^{(1)}}{r} \right) \left(\frac{m'}{\sin\theta} \frac{dP_l^m}{d\theta} P_{l'}^{m'} - \frac{m}{\sin\theta} P_l^m \frac{dP_{l'}^{m'}}{d\theta} \right) \Big[2\pi \delta_{(-m)m'} \Big] e^{-2i\omega t}$$

$$+ ik g_{lm} g_{l'm'} \left(\frac{dh_l^{(1)}}{dr} + \frac{h_l^{(1)}}{r} \right) h_{l'}^{(1)} \left(\frac{dP_l^m}{d\theta} \frac{dP_{l'}^{m'}}{d\theta} - \frac{mm'}{\sin^2\theta} P_l^m P_{l'}^{m'} \right) \Big[2\pi \delta_{(-m)m'} \Big] e^{-2i\omega t}$$

$$+ ik d_{lm} d_{l'm'} h_l^{(1)} \left(\frac{dh_{l'}^{(1)}}{dr} + \frac{h_{l'}^{(1)}}{r} \right) \left(\frac{-mm'}{\sin^2\theta} P_l^m P_{l'}^{m'} + \frac{dP_l^m}{d\theta} \frac{dP_{l'}^{m'}}{d\theta} \right) \Big[2\pi \delta_{(-m)m'} \Big] e^{-2i\omega t}$$

$$+ ik^2 d_{lm} g_{l'm'} h_l^{(1)} h_{l'}^{(1)} \left(\frac{-m}{\sin\theta} P_l^m \frac{dP_{l'}^{m'}}{d\theta} + \frac{m'}{\sin\theta} \frac{dP_l^m}{d\theta} P_{l'}^{m'} \right) \Big[2\pi \delta_{(-m)m'} \Big] e^{-2i\omega t}$$

$$+ i g_{lm} d_{l'm'}^{*} \left(\frac{dh_l^{(1)}}{dr} + \frac{h_l^{(1)}}{r} \right) \left(\frac{dh_{l'}^{(2)}}{dr} + \frac{h_{l'}^{(2)}}{r} \right) \left(\frac{-m'}{\sin\theta} \frac{dP_l^m}{d\theta} P_{l'}^{m'} - \frac{m}{\sin\theta} P_l^m \frac{dP_{l'}^{m'}}{d\theta} \right) \Big[2\pi \delta_{mm'} \Big]$$

$$+ ik g_{lm} g_{l'm'}^{*} \left(\frac{dh_l^{(1)}}{dr} + \frac{h_l^{(1)}}{r} \right) h_{l'}^{(2)} \left(-\frac{dP_l^m}{d\theta} \frac{dP_{l'}^{m'}}{d\theta} - \frac{mm'}{\sin^2\theta} P_l^m P_{l'}^{m'} \right) \Big[2\pi \delta_{mm'} \Big]$$

$$+ ik d_{lm} d_{l'm'}^{*} h_l^{(1)} \left(\frac{dh_{l'}^{(2)}}{dr} + \frac{h_{l'}^{(2)}}{r} \right) \left(\frac{mm'}{\sin^2\theta} P_l^m P_{l'}^{m'} + \frac{dP_l^m}{d\theta} \frac{dP_{l'}^{m'}}{d\theta} \right) \Big[2\pi \delta_{mm'} \Big]$$

$$+ ik^2 d_{lm} g_{l'm'}^{*} h_l^{(1)} h_{l'}^{(2)} \left(\frac{m}{\sin\theta} P_l^m \frac{dP_{l'}^{m'}}{d\theta} + \frac{m'}{\sin\theta} \frac{dP_l^m}{d\theta} P_{l'}^{m'} \right) \Big[2\pi \delta_{mm'} \Big]$$

$$+ c.c. \Bigg)$$

Perform the indicated summation over m' to obtain:

$$\frac{1}{c}W_U = \frac{1}{8}\int_0^{\pi}\sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}\sum_{m=-l}^{l} r^2\sin\theta\, d\theta \cdot \tag{VI.9$'$}$$

$$\cdot\Bigg(-ig_{lm}\, d_{l'(-m)}\Big(\frac{dh_l^{(1)}}{dr}+\frac{h_l^{(1)}}{r}\Big)\Big(\frac{dh_{l'}^{(1)}}{dr}+\frac{h_{l'}^{(1)}}{r}\Big)\Big(\frac{m}{\sin\theta}\frac{dP_l^m}{d\theta}P_{l'}^m + \frac{m}{\sin\theta}P_l^m\frac{dP_{l'}^{-m}}{d\theta}\Big)e^{-2i\omega t}$$

$$+ ikg_{lm}\, g_{l'(-m)}\Big(\frac{dh_l^{(1)}}{dr}+\frac{h_l^{(1)}}{r}\Big)h_{l'}^{(1)}\Big(\frac{dP_l^m}{d\theta}\frac{dP_{l'}^{-m}}{d\theta} + \frac{m^2}{\sin^2\theta}P_l^m P_{l'}^{-m}\Big)e^{-2i\omega t}$$

$$+ ikd_{lm}\, d_{l'(-m)}h_l^{(1)}\Big(\frac{dh_{l'}^{(1)}}{dr}+\frac{h_{l'}^{(1)}}{r}\Big)\Big(\frac{m^2}{\sin^2\theta}P_l^m P_{l'}^{m'} + \frac{dP_l^m}{d\theta}\frac{dP_{l'}^{-m}}{d\theta}\Big)e^{-2i\omega t}$$

$$- ik^2 d_{lm}\, g_{l'(-m)}h_l^{(1)}h_{l'}^{(1)}\Big(\frac{m}{\sin\theta}P_l^m\frac{dP_{l'}^{-m}}{d\theta} + \frac{m}{\sin\theta}\frac{dP_l^m}{d\theta}P_{l'}^{-m}\Big)e^{-2i\omega t}$$

$$- ig_{lm}\, d_{l'm}^{*}\Big(\frac{dh_l^{(1)}}{dr}+\frac{h_l^{(1)}}{r}\Big)\Big(\frac{dh_{l'}^{(2)}}{dr}+\frac{h_{l'}^{(2)}}{r}\Big)\Big(\frac{m}{\sin\theta}\frac{dP_l^m}{d\theta}P_{l'}^m + \frac{m}{\sin\theta}P_l^m\frac{dP_{l'}^m}{d\theta}\Big)$$

$$- ikg_{lm}\, g_{l'm}^{*}\Big(\frac{dh_l^{(1)}}{dr}+\frac{h_l^{(1)}}{r}\Big)h_{l'}^{(2)}\Big(\frac{dP_l^m}{d\theta}\frac{dP_{l'}^m}{d\theta} + \frac{m^2}{\sin^2\theta}P_l^m P_{l'}^m\Big)$$

$$+ ikd_{lm}\, d_{l'm}^{*}h_l^{(1)}\Big(\frac{dh_{l'}^{(2)}}{dr}+\frac{h_{l'}^{(2)}}{r}\Big)\Big(\frac{m^2}{\sin^2\theta}P_l^m P_{l'}^m + \frac{dP_l^m}{d\theta}\frac{dP_{l'}^m}{d\theta}\Big)$$

$$+ ik^2 d_{lm}\, g_{l'm}^{*}h_l^{(1)}h_{l'}^{(2)}\Big(\frac{m}{\sin\theta}P_l^m\frac{dP_{l'}^m}{d\theta} + \frac{m}{\sin\theta}\frac{dP_l^m}{d\theta}P_{l'}^m\Big)$$

$$+ \text{c.c.}\Bigg)$$

Use equations (App.C2.3) and (App.C3.3 and 4) to obtain:

$$\frac{1}{c}W_U = \frac{1}{8}\sum_{l=0}^{\infty}\sum_{l'=0}^{\infty}\sum_{m=-l}^{l} r^2 \cdot \tag{VI.9$'$}$$

$$\cdot\Big(- ig_{lm}\, d_{l'(-m)}\Big(\frac{dh_l^{(1)}}{dr}+\frac{h_l^{(1)}}{r}\Big)\Big(\frac{dh_{l'}^{(1)}}{dr}+\frac{h_{l'}^{(1)}}{r}\Big)\Big[\text{ Zero }\Big]e^{-2i\omega t}$$

$$+ ikg_{lm}\, g_{l'(-m)}\Big(\frac{dh_l^{(1)}}{dr}+\frac{h_l^{(1)}}{r}\Big)h_{l'}^{(1)}\Big[\frac{2l(l+1)}{2l+1}(-1)^m\delta_{ll'}\Big]e^{-2i\omega t}$$

$$+ ikd_{lm}\, d_{l'(-m)}h_l^{(1)}\Big(\frac{dh_{l'}^{(1)}}{dr}+\frac{h_{l'}^{(1)}}{r}\Big)\Big[\frac{2l(l+1)}{2l+1}(-1)^m\delta_{ll'}\Big]e^{-2i\omega t}$$

$$- ik^2 d_{lm}\, g_{l'(-m)}h_l^{(1)}h_{l'}^{(1)}\Big[\text{ Zero }\Big]e^{-2i\omega t}$$

$$- ig_{lm}\, d^*_{l'm}\Big(\frac{dh_l^{(1)}}{dr}+\frac{h_l^{(1)}}{r}\Big)\Big(\frac{dh_{l'}^{(2)}}{dr}+\frac{h_{l'}^{(2)}}{r}\Big)\Big[\text{ Zero }\Big]$$

$$- ikg_{lm}\, g^*_{l'm}\Big(\frac{dh_l^{(1)}}{dr}+\frac{h_l^{(1)}}{r}\Big)h_{l'}^{(2)}\Big[\frac{2l(l+1)}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'}\Big]$$

$$+ ikd_{lm}\, d^*_{l'm}h_l^{(1)}\Big(\frac{dh_{l'}^{(2)}}{dr}+\frac{h_{l'}^{(2)}}{r}\Big)\Big[\frac{2l(l+1)}{2l+1}\frac{(l+m)!}{(l-m)!}\delta_{ll'}\Big]$$

$$+ ik^2 d_{lm}\, g^*_{l'm}h_l^{(1)}h_{l'}^{(2)}\Big[\text{ Zero }\Big]$$

$$+ \ c.c.\Big)$$

Perform the indicated summation over l' to obtain:

$$\frac{1}{c}W_U = \frac{1}{4}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l(l+1)}{(2l+1)}(-1)^m r^2 \begin{pmatrix} ikg_{lm}\,g_{l(-m)}\left(\frac{dh_l^{(1)}}{dr}+\frac{h_l^{(1)}}{r}\right)h_l^{(1)} \\ +ikd_{lm}\,d_{l(-m)}h_l^{(1)}\left(\frac{dh_l^{(1)}}{dr}+\frac{h_l^{(1)}}{r}\right)\end{pmatrix} e^{-2i\omega t}$$

$$+\ \frac{1}{4}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l(l+1)}{(2l+1)}\frac{(l+m)!}{(l-m)!}r^2 \begin{pmatrix} -ikg_{lm}\,g_{lm}^{*}\left(\frac{dh_l^{(1)}}{dr}+\frac{h_l^{(1)}}{r}\right)h_l^{(2)} \\ +ikd_{lm}\,d_{lm}^{*}h_l^{(1)}\left(\frac{dh_l^{(2)}}{dr}+\frac{h_l^{(2)}}{r}\right)\end{pmatrix}$$

$$+\ c.c. \qquad \text{(VI.9}'\text{)}$$

Combine terms:

$$\frac{1}{c}W_U = \frac{1}{4}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l(l+1)}{(2l+1)}(-1)^m\left(g_{lm}\,g_{l(-m)} + d_{lm}\,d_{l(-m)}\right)\cdot$$

$$\cdot\left(ikr^2\right)h_l^{(1)}\left(\frac{dh_l^{(1)}}{dr}+\frac{h_l^{(1)}}{r}\right)e^{-2i\omega t} + c.c.$$

$$+\ \frac{1}{4}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l(l+1)}{(2l+1)}\frac{(l+m)!}{(l-m)!}\left(g_{lm}\,g_{lm}^{*} + d_{lm}\,d_{lm}^{*}\right)\cdot$$

$$\cdot\left(-ikr^2\right)\left(\left(\frac{dh_l^{(1)}}{dr}+\frac{h_l^{(1)}}{r}\right)h_l^{(2)} - h_l^{(1)}\left(\frac{dh_l^{(2)}}{dr}+\frac{h_l^{(2)}}{r}\right)\right)$$

Use equations (App.B2.8) and (App.B2.6):

$$\frac{1}{c}W_U = \frac{ik}{4}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l(l+1)}{(2l+1)}(-1)^m\left(g_{lm}\,g_{l(-m))} + d_{lm}\,d_{l(-m)}\right)\cdot$$

$$\cdot\, r^2 h_l^{(1)}\left((l+1)\frac{h_l^{(1)}}{r} - k h_{l+1}^{(1)}\right)e^{-2i\omega t} + c.c.$$

$$+ \frac{1}{4}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l(l+1)}{(2l+1)}\frac{(l+m)!}{(l-m)!}\left(g_{lm}\,g_{lm}^{*} + d_{lm}\,d_{lm}^{*}\right)\left(-ikr^2\right)\left[\frac{2i}{kr^2}\right]$$

Re-arrange terms to obtain:

$$\frac{1}{c}W_U = -\frac{k^2r^2}{4}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l(l+1)}{(2l+1)}(-1)^m e^{-2i\omega t} h_l^{(1)} h_{l+1}^{(1)}\left(g_{lm}\,g_{l(-m))} + d_{lm}\,d_{l(-m)}\right)$$

$$+ \quad c.c.$$

$$+ \frac{ikr}{4}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l(l+1)^2}{(2l+1)}(-1)^m e^{-2i\omega t} h_l^{(1)} h_l^{(1)}\left(g_{lm}\,g_{l(-m))} + d_{lm}\,d_{l(-m)}\right)$$

$$+ \quad c.c.$$

$$+ \frac{1}{2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l(l+1)}{(2l+1)}\frac{(l+m)!}{(l-m)!}\left(g_{lm}\,g_{lm}^{*} + d_{lm}\,d_{lm}^{*}\right)$$

In actual applications, the two frequency-dependent terms in the above expression are generally not of interest. The essential physical content of the problem lies exclusively in the *dc* component, *i.e.*, the third term. Therefore, it is typical to calculate the ***time-averaged*** energy flux. From the above, one would obtain:

Time—Averaged Electromagnetic Energy Flux: (VI.10)

$$\frac{1}{c}\left\langle W_U \right\rangle \;=\; \frac{1}{2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l(l+1)}{(2l+1)}\frac{(l+m)!}{(l-m)!}\left(g_{lm}\,g_{lm}^{*} \;+\; d_{lm}\,d_{lm}^{*}\right)$$

Note the positive-definite nature of this expression. Note also the complete absence of cross-terms within it; the only paired-combination of g_{lm} and d_{lm} coefficients to appear are those in which the term multiplies itself, and no other.

The initial objective to express radiated energy flux entirely in terms of known system parameters has thus been accomplished. The "known" parameters in this case are the system electric and magnetic multipole moments, g_{lm} and d_{lm}, respectively.

Similar calculations can be performed for the other conserved quantities. The results will be merely quoted, the derivations being too lengthy to include here. The relevant fluxes are given in their time-averaged form. Note that for the most part, a small number of cross-terms appear in each expression, but never more than a few. The overwhelming majority of cross terms integrate out of the final expression as a result of orthogonality properties among the two sets of angular functions.

Electromagnetic Momentum Flux (x-component): (VI.11)

$$\left\langle W_{Gx} \right\rangle = \frac{i}{4} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \frac{1}{(2l+1)} \frac{(l+m)!}{(l-m)!} \cdot$$
$$\cdot \left(\begin{array}{l} \frac{l(l+2)}{(2l+3)} \left(g_{(l+1)(m-1)} g_{lm}^{*} + d_{(l+1)(m-1)} d_{lm}^{*} \right) \\ + \frac{(l-1)(l+1)}{(2l-1)} \left(g_{(l-1)(m-1)} g_{lm}^{*} + d_{(l-1)(m-1)} d_{lm}^{*} \right) \\ + \left(g_{l(m-1)} d_{lm}^{*} - d_{l(m-1)} g_{lm}^{*} \right) \end{array} \right) + c.c.$$

Electromagnetic Momentum Flux (y-component): (VI.12)

$$\left\langle W_{Gy} \right\rangle = -\frac{1}{4} \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \frac{1}{(2l+1)} \frac{(l+m)!}{(l-m)!} \cdot$$
$$\cdot \left(\begin{array}{l} \frac{l(l+2)}{(2l+3)} \left(g_{(l+1)(m-1)} g_{lm}^{*} + d_{(l+1)(m-1)} d_{lm}^{*} \right) \\ + \frac{(l-1)(l+1)}{(2l-1)} \left(g_{(l-1)(m-1)} g_{lm}^{*} + d_{(l-1)(m-1)} d_{lm}^{*} \right) \\ + \left(g_{l(m-1)} d_{lm}^{*} - d_{l(m-1)} g_{lm}^{*} \right) \end{array} \right) + c.c.$$

Electromagnetic Momentum Flux (z-component): (VI.13)

$$\left\langle W_{Gz}\right\rangle = -\frac{i}{4}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\frac{1}{(2l+1)}\frac{(l+m)!}{(l-m)!}\cdot$$

$$\cdot\left(\begin{array}{l} \frac{l(l+2)(l+m+1)}{(2l+3)}\left(g_{(l+1)m}\,g_{lm}^{*} + d_{(l+1)m}\,d_{lm}^{*}\right) \\ -\frac{(l-1)(l+1)(l-m)}{(2l-1)}\left(g_{(l-1)m}\,g_{lm}^{*} + d_{(l-1)m}\,d_{lm}^{*}\right) \\ + \quad m\left(g_{lm}\,d_{lm}^{*} - d_{lm}\,g_{lm}^{*}\right) \end{array}\right) + c.c.$$

Electromagnetic Angular Momentum Flux (x-component): (VI.14)

$$k\left\langle W_{Mx}\right\rangle = \frac{1}{4}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l(l+1)}{(2l+1)}\frac{(l+m)!}{(l-m)!}\cdot$$

$$\cdot\left(g_{l(m-1)}g_{lm}^{*} + g_{l(m-1)}^{*}g_{lm} + d_{l(m-1)}d_{lm}^{*} + d_{l(m-1)}^{*}d_{lm}\right)$$

Electromagnetic Angular Momentum Flux (y-component): (VI.15)

$$k\left\langle W_{My}\right\rangle = -\frac{i}{4}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l(l+1)}{(2l+1)}\frac{(l+m)!}{(l-m)!}\cdot$$

$$\cdot\left(g_{l(m-1)}g_{lm}^{*} - g_{l(m-1)}^{*}g_{lm} + d_{l(m-1)}d_{lm}^{*} - d_{l(m-1)}^{*}d_{lm}\right)$$

Electromagnetic Angular Momentum Flux (z-component): (VI.16)

$$k\left\langle W_{Mz}\right\rangle = \frac{1}{2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l(l+1)}{(2l+1)}\frac{(l+m)!}{(l-m)!}\,m\left(g_{lm}\,g_{lm}^{*} + d_{lm}\,d_{lm}^{*}\right)$$

Before proceeding, it is beneficial to examine the above seven expressions in somewhat greater detail.

It was mentioned earlier that the particular geometry used to calculate the above flux quantities, namely two concentric spheres of inner and outer radius R_1 and R_2, was actually more comprehensive than might be initially supposed. Just how generic this particular choice of geometry actually is for problems of this sort will be made clear in what follows.

Recall once again the general derivation of conservation laws. We start with a relation of the form:

$$-\frac{1}{c}\frac{\partial a^0}{\partial t} = \nabla\cdot\vec{a}$$

Integrate over a suitable volume V:

$$-\frac{1}{c}\frac{\partial}{\partial t}\int_V a^0\,dV = \int_V \nabla\cdot\vec{a}\,dV$$

Use Gauss's Divergence Theorem on the R.H.S.:

$$-\frac{1}{c}\frac{\partial A^0}{\partial t} = \oint_S \vec{a}\cdot\hat{n}\,dS$$

For any volume V subtended by inner Gaussian surface S_1 (not necessarily a sphere) and outer Gaussian surface S_2 (also not necessarily a sphere), we have:

$$-\frac{1}{c}\frac{\partial A^0}{\partial t} = \oint_{S_2} \vec{a}\cdot\hat{n}\,dS_2 - \oint_{S_1} \vec{a}\cdot\hat{n}\,dS_1$$

NOTE: By "Gaussian surface", one means any closed, bounded, simply-connected 2-D surface. (Tori and other multiply-connected domains do not qualify.)

Take time-averages of the above quantities:

$$\left\langle -\frac{1}{c}\frac{\partial A^0}{\partial t} \right\rangle = \left\langle \oint_{S_2} \vec{a}\cdot\hat{n}\, dS_2 \right\rangle - \left\langle \oint_{S_1} \vec{a}\cdot\hat{n}\, dS_1 \right\rangle \qquad \text{(VI.17)}$$

We are now in a position to exploit the particular properties of the monochromatic solution, (III.F.7 thru 16).

The $\vec{E}$ and $\vec{B}$ quantities display a first order time dependence on $e^{\pm i\omega t}$. The seven conserved quantities of (VI.10 thru 16) all depend quadratically on $\vec{E}$ and $\vec{B}$, *cf.,* (V.A.6 thru 8). They therefore display a second-order dependence on $e^{\pm i\omega t}$ of the form:

$$A^0 = A_n e^{-2i\omega t} + A_{dc} + A_p e^{2i\omega t} \qquad \text{(VI.18)}$$

where the A_n, A_{dc}, and A_p terms are *time-independent* double-summations over the indices l and m. (If any of the four terms of the auxiliary vector (F,G,H,I) had contained time-dependent terms, (see equations V.D.4 thru V.D.11) we would not be able to assert that the A_n, A_{dc}, and A_p are, in fact, time-indpendent.) The explicit expressions for A_n, A_{dc}, and A_p have been presented in equations (VI.10 thru 16) and need not concern us here. Merely knowing that they are time-independent is sufficient to complete this discussion.

We easily calculate from (VI.18) that:

$$-\frac{1}{c}\frac{\partial A^0}{\partial t} = 2ikA_n e^{-2i\omega t} - 2ikA_p e^{2i\omega t} \tag{VI.19}$$

Since:

$$\left\langle e^{\pm i\omega t}\right\rangle = 0 \tag{VI.20}$$

We derive that:

$$\left\langle -\frac{1}{c}\frac{\partial A^0}{\partial t}\right\rangle = 0 \tag{VI.21}$$

Thus, the time-averaged A^0 behavior is steady-state, as it intuitively should be for conserved quantities.

Consequently, we have:

$$\left\langle \oint_{S_2} \vec{a}\cdot\hat{n}\, dS_2 \right\rangle = \left\langle \oint_{S_1} \vec{a}\cdot\hat{n}\, dS_1 \right\rangle \tag{VI.22}$$

The above is the time-averaged equivalent of the law claimed for the concentric sphere case, (VI.7):

$$\left[W_{A^0}\right]_{R_2} = \left[W_{A^0}\right]_{R_1}$$

but generalized to *arbitrary* Gaussian surfaces (subject to one modest constraint to be mentioned below).

This remarkable generalization to arbitrarily-shaped surfaces follows inexorably from the simplest of properties of the monochromatic Maxwell solutions, namely, (VI.18). It allows one to calculate outward fluxes from radiation sources whose surfaces are seemingly much be too complicated to treat mathematically. The flux calculation from some vilely deformed Gaussian surface S_1 becomes entirely possible by calculating it instead on some "simple" surface S_2 that completely encloses surface S_1.

In practical terms, the method would work out as follows. Calculate the general flux expression in terms of the field expansion coefficients g_{lm} and d_{lm} on the outlying "simple" surface S_2. (This has already been done done in equations (VI.10 thru 16)). Then use boundary conditions imposed on the problem by the deformed inner surface S_1 to determine the specific analytical expressions for g_{lm} and d_{lm}. Thus, the problem is solved completely, without recourse to surface integration over the "difficult" surface S_1.

It is precisely this property that is to be exploited in applications to be discussed in a separate report. This delightful feature of Maxwellian electrodynamics would have gone entirely unnoticed if the original radiation problem had been solved in Cartesian coordinates. Spherical coordinates highlight many such hidden features of the Maxwellian formulation.

It was mentioned above that there was one minor constraint on the selection of inner Gaussian surface S_1. It turns out that the $\vec{E}$ and $\vec{B}$ solutions given in Chapter III of this report are singular at the origin. (The spherical Hankel functions "blow-up" at the point $r=0$.) Therefore, for all the mathematical steps to hold, we require that the Gaussian surfaces S_1 and S_2 *both enclose the* *origin.* This represents only a minor restriction in actual applications, but it is an aspect that should not be overlooked when setting up problems of this sort.

Stated in full generality, with all the attendant provisos and restrictions, we say that the steady-state flux of conserved electromagnetic quantities from monochromatic sources, into and out of source-free regions of space subtended by Gaussian surfaces S_1 and S_2 that both enclose the origin, must be equal. "Source-free" implies that $(c\rho, \vec{J}) = 0$ within the volume V; hence, the present formulation would not be suitable for rf-plasmas. Also, the monochromatic solution is inherently steady-state; thus, we do not consider system transients or propagation delays in this formulation. Lastly, "into" volume V means propagation inward through inner Gaussian surface S_1; "out of" volume V means propagation outward through outer Gaussian surface S_2.

The intuitive notion underlying all this mathematics is the following. The transport of the seven quantities examined above can be compared to a steady-state fluid flow of incompressible fluid from a point source. For a source-free, sink-free volume V that is bounded on the inside by Gaussian surfce S_1 and on the outside by Gaussian surface S_2, both of which enclose the point source, the amount of incoming flow through S_1 must exactly equal the amount of outgoing flow through S_2. The fluid can be neither compressed, consumed, or generated within volume V.

These flux flow relations also serve as the 3-D equivalent of Cauchy's line integral theorem of complex analysis. In the 2-D case of complex functions, certain conditions on the function $F(z)$ assure that the integral of $F(z)$ on a closed 2-D path will be independent of the path chosen. The analogy to the 3-D case is embodied in formula (VI.22). Schematically, we would have Figure 1.

It should also be mentioned that the flux law, especially as stated in the form immediately preceding equation (VI.17), is the 3-D analog of the Fundamental Theorem of (one-dimensional) Calculus, namely,

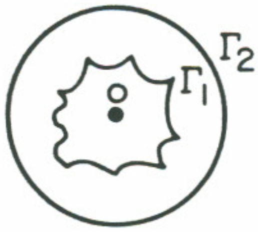
$$\int_V \nabla \cdot \vec{a}\, dV \;=\; \oint_{S_2} \vec{a} \cdot \hat{n}\, dS_2 \;-\; \oint_{S_1} \vec{a} \cdot \hat{n}\, dS_1 \qquad \text{(VI.23)}$$

versus

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial x}\, dx \;=\; \Big[F\Big]_{x_2} - \Big[F\Big]_{x_1} \qquad \text{(VI.24)}$$

Both the above laws express the fact that an integral over a selected region of N-dimensional space can be expressed in terms of integrals over the (N-1)-dimensional subspaces that bound the original N-dimensional space.

2-D Case:

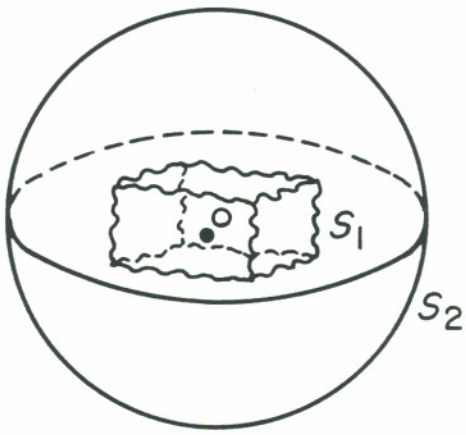


- Jordan contours Γ_1, Γ_2 simply-connected
- No poles in region bounded by Γ_1 and Γ_2
- Cauchy-Riemann equations for u, v, where $f = u + iv$

Then:

$$\oint_{\Gamma_1} f(z)\, dz_1 = \oint_{\Gamma_2} f(z)\, dz_2$$

3-D Case:



- Gaussian Surfaces S_1 and S_2 simply-connected
- No sources, sinks in region bounded by S_1 and S_2
- Divergeneceless vector $\vec{a}$

Then:

$$\oint_{S_1} \vec{a} \cdot \hat{n}\, dS_1 = \oint_{S_2} \vec{a} \cdot \hat{n}\, dS_2$$

FIGURE 1

A Comparison of Flux Integrals in Two and Three dimensions

CHAPTER VII
QUANTUM ASPECTS

Consider the time-averaged fluxes for energy and z-component of angular momentum:

$$\left\langle W_U \right\rangle = \frac{c}{2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l(l+1)}{(2l+1)}\frac{(l+m)!}{(l-m)!}\left(g_{lm}\,g_{lm}^{*} + d_{lm}\,d_{lm}^{*}\right) \qquad \text{(VII.1)}$$

$$\left\langle W_{Mz} \right\rangle = \frac{1}{2k}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l(l+1)}{(2l+1)}\frac{(l+m)!}{(l-m)!}\,m\left(g_{lm}\,g_{lm}^{*} + d_{lm}\,d_{lm}^{*}\right) \qquad \text{(VII.2)}$$

For the moment, disregard the two summations over l and m in order to key in on the energy and z-component of angular momentum for a single mode:

$$\left\langle W_U \right\rangle_{lm} = \frac{c}{2}\frac{l(l+1)}{(2l+1)}\frac{(l+m)!}{(l-m)!}\left(g_{lm}\,g_{lm}^{*} + d_{lm}\,d_{lm}^{*}\right) \qquad \text{(VII.3)}$$

$$\left\langle W_{Mz} \right\rangle_{lm} = \frac{1}{2k}\frac{l(l+1)}{(2l+1)}\frac{(l+m)!}{(l-m)!}\,m\left(g_{lm}\,g_{lm}^{*} + d_{lm}\,d_{lm}^{*}\right) \qquad \text{(VII.4)}$$

At this point, invoke Planck's energy formula for single-mode radiation as inferred from his analysis of black body emission spectra:

$$\left\langle U \right\rangle_{lm} = \left\langle W_U \right\rangle_{lm}\Delta\tau = \hbar\omega \qquad \text{(VII.5)}$$

$\Delta\tau$ is a constant with units of time. It can be shown to be equal to $(2\omega)^{-1}$, but rather than committing several additional paragraphs to demonstrate this, I prefer instead to leave it in the raw form $\Delta\tau$ and advance on to the essential points of this discussion.

If this formula for single-mode energy is true, it *forces* the radiated z-component of angular momentum to be:

$$\left\langle Mz \right\rangle_{lm} = \left\langle W_{Mz} \right\rangle_{lm} \Delta\tau = m\hbar \qquad \text{(VII.6)}$$

as can be quickly ascertained by plugging (VII.3) into (VII.4).

This, it turns out, is a familiar formula from quantum mechanics, but rarely is it demonstrated how it follows from classical principles. One consequence is that it allows one to treat the single radiation mode as an entity with well-defined energy and z-component of angular momentum.

But this is not all. The general formulas for W_U and W_{Mz}, (VII.1 & 2), contain *no* cross terms in them, the summations being over g_{lm} and d_{lm} terms times their own complex conjugates (and none other). Thus, from an energy or z-angular momentum standpoint, there is nothing to prevent treating the full radiation field as an ensemble of individual particles, each with its own characteristic energy and z-angular momentum. The particle nature of the ensemble is embodied in the fact that the energy of one lm mode in no way depends on the energy of some other mode. There is no wave-like "interference" among elements of this ensemble.

This characteristic, in fact, is what might have prompted Einstein to postulate the photon concept in the first place.

One of the more interesting consequences of (VII.3 and 4) is the following. The lm-mode can be considered as a sub-species of the more generic l-mode. The energy and z-angular momentum of the l-mode are obtained by summing (VII.3) and (VII.4) over the index m, leaving l fixed. The ensemble of l-modes is interesting in its own right, as will be demonstrated subsequently.

In a thermodynamically equilibrated ensemble of l-modes, one would expect an equipartition of energy among them. In line with "classical" thermodynamic reasoning, one would consider this equipartitioned amount of energy to be proportional to temperature. In terms of Boltzmann's constant k_B, one would stipulate the l-mode micro-state energy to be:

$$\sum_{m=-l}^{l} \frac{c}{2} \frac{l(l+1)}{(2l+1)} \frac{(l+m)!}{(l-m)!} \left(g_{lm}\, g_{lm}^{*} \;+\; d_{lm}\, d_{lm}^{*} \right) \Delta\tau \;=\; k_B T \qquad \text{(VII.7)}$$

However, in the context of monochromatic waves, which is what is being examined here, this microstate energy would be better stipulated as being equipartioned among the sub-specie lm-modes and proportional to *frequency* rather than temperature. Hence, in keeping with the experimental observations of Planck, one would have:

$$\frac{c}{2} \frac{l(l+1)}{(2l+1)} \frac{(l+m)!}{(l-m)!}\, g_{lm}\, g_{lm}^{*}\, \Delta\tau \;=\; \frac{c}{2} \frac{l(l+1)}{(2l+1)} \frac{(l+m)!}{(l-m)!}\, d_{lm}\, d_{lm}^{*}\, \Delta\tau \;=\; \frac{1}{4}\hbar\omega \quad \text{(VII.8)}$$

The pre-factor of 1/4 is selected with foreknowledge of the final outcome. Final formulas become slightly inconvenient with this factor missing at this stage of the derivation. The primary proportionality constant, $\hbar$, is simply Planck's constant h divided by 2π.

Thus, from (VII.3):

$$\left\langle U \right\rangle_{lm} = \frac{1}{2}\hbar\omega \tag{VII.9}$$

From which one obtains for the l-mode:

$$\begin{aligned} \left\langle U \right\rangle_{l} &= \sum_{m=-l}^{l} \left\langle U \right\rangle_{lm} \\ &= \sum_{m=-l}^{l} \frac{1}{2}\hbar\omega \\ &= \frac{1}{2}(2l+1)\hbar\omega \\ &= (l + ½)\hbar\omega \end{aligned} \tag{VII.10}$$

This is the familiar energy formula for the linear harmonic oscillator of quantum mechanics, a pivotal quantity in the Schrödinger explanation of the blackbody emission spectrum.

The corresponding l-mode z-angular momentum is:

$$\begin{aligned} \left\langle Mz \right\rangle_{l} &= \sum_{m=-l}^{l} \left\langle Mz \right\rangle_{lm} \\ &= \sum_{m=-l}^{l} \frac{m}{2}\hbar\omega \\ &= [\text{Zero}] \end{aligned} \tag{VII.11}$$

Thus the l-mode "particle" carries with it an energy given by (VII.10) and a z-component of angular momentum of zero, as given by (VII.11).

One now has in hand enough knowledge about radiative l-modes to embark on a full-blown re-derivation of Planck's radiation law within the framework of the electrodynamic formalism. The discussion from this point onward relies heavily on the methods of statistical mechanics. Many applicable formulas will be invoked without prior derivation. A familiarity with the statistical mechanical approach is therefore assumed of the reader for the remaining portions of this report.

Recall that each l-mode microstate, when present, would possess $(l+½)\hbar\omega$ units of energy. The total energy of the ensemble would be the sum of these individual microstate energies. However, not all microstates of the ensemble are "active". Hence, the total energy is not a simple sum over l-mode energies but rather a weighted sum, the weight-factors indicating which percentage of the l-mode microstates are "active" and which are "dormant". Imposition of macroscopic (thermodynamic) constraints on the system are sufficient to completely determine the appropriate weight-factors for each microstate. The thermodynamic requirement that the total energy of the system be a fixed constant forces the weight-factors to assume the familiar Boltzmann form, $\exp(-E_l/k_B T)$, where E_l is the energy associated with the l-microstate, T is the temperature, and k_B is Boltzmann's constant. For the l-mode described above, this works out to be $\exp(-(l+½)\hbar\omega/k_B T)$.

The average energy per radiation mode would be calculated according to familiar statistical rules:

$$\left\langle E \right\rangle = \textit{average energy per } l\textit{-mode}$$

$$= \left(\frac{\textit{Sum of active } l\textit{-mode energies}}{\textit{Sum of active } l\textit{-modes}} \right)$$

$$= \left(\frac{\sum_{l=0}^{\infty} (l+½)\hbar\omega \, e^{-(l+½)\hbar\omega/k_B T}}{\sum_{l=0}^{\infty} e^{-(l+½)\hbar\omega/k_B T}} \right)$$

$$= \left(\frac{\sum_{l=0}^{\infty} (l+½)\hbar\omega \, e^{-l\hbar\omega/k_B T}}{\sum_{l=0}^{\infty} e^{-l\hbar\omega/k_B T}} \right)$$

$$= \left(\frac{\hbar\omega}{e^{\hbar\omega/k_B T} - 1} + ½\hbar\omega \right) \qquad \text{(VII.12)}$$

Note an interesting thing here. In the limit that $\hbar$ approaches zero, the above average-energy expression approaches $k_B T$, the very value that would have been stipulated in the "classical" formulation, *cf.* (VII.7). Therefore, it is evident that Planck's quantum formulation properly reduces down to the classical formulation in the limit that $\hbar$ approaches zero, as indeed any quantum theory is required to.

With thermally-dependent weight-factors properly accounted for, one obtains alternate expressions for the expansion coefficients g_{lm} and d_{lm}. In place of (VII.8), one requires:

$$\frac{c}{2}\frac{l(l+1)}{(2l+1)}\frac{(l+m)!}{(l-m)!}g_{lm}\,g_{lm}^{*}\,\Delta\tau \;=\; \frac{c}{2}\frac{l(l+1)}{(2l+1)}\frac{(l+m)!}{(l-m)!}d_{lm}\,d_{lm}^{*}\,\Delta\tau \;= \qquad \text{(VII.13)}$$

$$= \frac{\hbar\omega}{4}\left(\frac{e^{-(l+1/2)\hbar\omega/k_BT}}{\sum_{l=0}^{\infty} e^{-(l+1/2)\hbar\omega/k_BT}}\right) = \frac{\hbar\omega}{4}\left(2\sinh\left(\frac{\hbar\omega}{2k_BT}\right)e^{-(l+1/2)\hbar\omega/k_BT}\right)$$

The essential thing to note here is the form of the weighting function for the lm-microstate energy. Multiplying the basic energy quantum of equation (VII.8) with the bracketted function of T in (VII.13) assures that the upcoming derivation of emission characteristics from the radiating black body will work out properly. The weighting function advocated by equation (VII.13) is neither the Boltzmann function $\exp(-E_l/k_BT)$, nor the Bose-Einstein function $\left((\exp(E_l/k_BT)\,)-1\right)^{-1}$, but rather something that falls midway between these two choices. The merits of this particular weighting function will be made clear in the developments of (VII.14).

All this aside, it is important to note what has been accomplished here. Except for the non-specification of $\Delta\tau$, an analytical expression for the magnitude of each expansion coefficient g_{lm} and d_{lm} has been supplied. These analytical expressions are then inserted into the general formulas for $\vec{\mathrm{E}}$ and $\vec{\mathrm{B}}$ as given in equations (I.23 thru 28). Since Maxwell's equations provide no restrictions on the analytical content of the expansion coefficients g_{lm} and d_{lm} other than their proscription against any dependence on position or time, *i.e.*, that they are forbidden to be functions of (r,θ,ϕ,t), and since the proposed analytical expressions of (VII.13) are functions solely of non-geometric parameters such as angular frequency ω and temperature T, they are perfectly acceptable candidates for solution. The consequent electric and magnetic fields yield energy fluxes that correspond to blackbody emission spectra, as will be demonstrated in the upcoming calculation of (VII.14). These thermally-dependent expansion coefficients for

$\vec{\mathrm{E}}$ and $\vec{\mathrm{B}}$ indicate that Maxwellian electrodynamics can assume thermodynamic character when the physical situation calls for it, and indicate exactly where the thermodynamic aspects must enter the theory. They also demonstrate that *vector* wave-functions can be invoked to explain blackbody emission spectra. These vector wave functions satisfy Maxwell's equations. (Scalar wave-functions would satisfy Schrödinger's equation.)

Enough groundwork has thus been established to perform actual calculations. The approach used here employs techniques that were developed after Planck's time; however, they rely heavily on his ingenious use of statistical principles and follow his example at all the key steps.

The energy density associated with an ensemble of radiating l-modes is given as the normalized six-dimensional integral over phase-space:

$$\int_{\nu} E(\nu,T)\,d\nu \;=\; \frac{1}{h^3}\iiint_{\text{p-space}} \left[\iiint_{\text{r-space}} \frac{\partial u}{\partial t}\,\Delta\tau\,dx\,dy\,dz\right] dp_x\,dp_y\,dp_z \qquad \text{(VII.14)}$$

where h^3 = unit-cell volume in plase space

u = electromagnetic energy density

Convert the integration to spherical coordinates:

$$= \frac{1}{h^3}\iiint_{\text{p-space}} \left[\int_0^{2\pi}\int_0^{\pi}\int_{R_1}^{R_2} \frac{\partial u}{\partial t}\,\Delta\tau\, r^2 \sin\theta\, dr\,d\theta\,d\phi\right] p^2 \sin\theta\, dp\,d\theta\,d\phi$$

Utilize equations (VI.9 and 10) for the energy density to obtain:

$$= \frac{1}{h^3}\iiint_{\text{p-space}} \left[\frac{c}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(E_\theta B_\phi - E_\phi B_\theta \right)\Delta\tau\, r^2 \sin\theta\, dr\, d\theta\, d\phi \right] p^2 \sin\theta\, dp\, d\theta\, d\phi$$

$$= \frac{1}{h^3}\iiint_{\text{p-space}} \left[\frac{c}{2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{l(l+1)}{(2l+1)}\frac{(l+m)!}{(l-m)!}\left(g_{lm}\, g_{lm}^* + d_{lm}\, d_{lm}^* \right)\Delta\tau \right] p^2 \sin\theta\, dp\, d\theta\, d\phi$$

The integration over p-space needs to converted to an integration over ν-space:

$$p^2 \sin\theta\, dp\, d\theta\, d\phi = \tag{VII.15}$$

$$= (\hbar k)^2 \sin\theta\, d(\hbar k)\, d\theta\, d\phi$$

$$= \hbar^3 k^2 \sin\theta\, dk\, d\theta\, d\phi$$

$$= \hbar^3 \left(\frac{\omega}{c}\right)^2 \sin\theta\, d\left(\frac{\omega}{c}\right) d\theta\, d\phi$$

$$= \frac{\hbar^3 \omega^2}{c^3} \sin\theta\, d\omega\, d\theta\, d\phi$$

$$= \frac{\left(\frac{h}{2\pi}\right)^3 (2\pi\nu)^2}{c^3} \sin\theta\, d(2\pi\nu)\, d\theta\, d\phi$$

$$= \frac{h^3 \nu^2}{c^3} \sin\theta\, d\nu\, d\theta\, d\phi$$

When this replacement is made in equation (VII.14), one obtains:

$$\int_\nu E(\nu,T)\, d\nu = \tag{VII.14$'$}$$

$$\int_0^{2\pi}\int_0^{\pi}\int_\nu \left[\frac{1}{2c^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{l(l+1)}{(2l+1)}\frac{(l+m)!}{(l-m)!}\left(g_{lm}\, g_{lm}^* + d_{lm}\, d_{lm}^* \right)\Delta\tau \right] \nu^2 \sin\theta\, d\nu\, d\theta\, d\phi$$

Perform the integration over angles:

$$= \int_\nu \left[\frac{2\pi}{c^2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{l(l+1)}{(2l+1)} \frac{(l+m)!}{(l-m)!} \left(g_{lm}\, g_{lm}^* \,+\, d_{lm}\, d_{lm}^* \right) \Delta\tau \right] \nu^2\, d\nu$$

$$= \int_\nu \frac{4\pi}{c^3} \left[\frac{c}{2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{l(l+1)}{(2l+1)} \frac{(l+m)!}{(l-m)!} \left(g_{lm}\, g_{lm}^* \,+\, d_{lm}\, d_{lm}^* \right) \Delta\tau \right] \nu^2\, d\nu$$

Invoke the (VII.13) relation:

$$= \int_\nu \frac{8\pi}{c^3} \left[\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{h\nu}{4} \left(2\sinh\!\left(\frac{\hbar\omega}{2k_BT}\right) e^{-(l+½)\hbar\omega/k_BT} \right) \right] \nu^2\, d\nu$$

Perform the summation over m:

$$= \int_\nu \frac{8\pi}{c^3} \left[\sum_{l=0}^{\infty} (2l+1) \frac{h\nu}{4} \left(2\sinh\!\left(\frac{\hbar\omega}{2k_BT}\right) e^{-(l+½)\hbar\omega/k_BT} \right) \right] \nu^2\, d\nu$$

Re-expand the $2\sinh(\hbar\omega/2k_BT)$ term as an infinite sum, *cf.*, (VII.13):

$$= \int_\nu \frac{8\pi}{c^3} \left[\left(\frac{\sum_{l=0}^{\infty} (l+½)\, h\nu\, e^{-(l+½)\, h\nu/k_BT}}{\sum_{l=0}^{\infty} e^{-(l+½)\, h\nu/k_BT}} \right) \right] \nu^2\, d\nu$$

Re-perform the summations of (VII.12):

$$= \int_\nu \frac{8\pi}{c^3} \left[\frac{h\nu}{e^{h\nu/k_BT} - 1} \,+\, ½ h\nu \right] \nu^2\, d\nu$$

Consider the integrand:

$$E(\nu,T) = \frac{8\pi h}{c^3}\left[\frac{\nu^3}{e^{h\nu/k_BT}-1} + \tfrac{1}{2}\nu^3\right] \qquad \text{(VII.16)}$$

The first term is Planck's analytical expression for blackbody emission as ascertained from experimental data collected by Lummer and Pringsheim[16], among others. It expresses the spectral energy density (Energy per volume per Hz) from a heated blackbody as a function of light frequency ν and temperature T. The second term is the so-called zero-point energy term, which represents the residual energy possessed by bodies at absolute zero. This term has interesting consequences in certain theoretical discussions, but in the context of blackbody radiation, is not important since it represents that portion of the energy reservoir that does not interact with the radiating modes.

The first term in (VII.16) is what is observed experimentally. It is imperative that any postulated configuration for the micro-states eventually infer this spectral distribution for the macro-state. Since the microstates as specified in (VII.13) lead to the required $E(\nu,T)$ formula, we can confidently assert that they are the correct choices for this physical situation. Note that Maxwell's equations are fully operative in this so-called "quantum regime". No inconsistency exists between the Maxwell energy expression and the experimentally obseved spectra of Planck, provided that the electromagnetic expansion coefficients g_{lm} and d_{lm} are selected according to the thermodynamic requirements of (VII.13). This runs counter to the frequently voiced opinion that the Planck hypothesis is incompatible with classical formulas. The Maxwell expressions for l-mode energy and z-angular momentum, as supplemented by Planck's hypothesis, (VII.5), and thermodynamic considerations, are perfectly adequate to explain the observed blackbody radiation characteristics.

The electrodynamic formulation also has the potential of providing a first-principles explanation for the Planck hypothesis, (VII.5). Preliminary mathematical investigations indicate that $\left\langle W_U \right\rangle$ is required to go as ω^2 to lowest order in ω. This in turn forces $\left\langle U \right\rangle$ to go as ω, which conforms nicely with Planck's "$\hbar\omega$" Energy Law. This aspect will be pursued in a future paper.

Planck's second hypothesis that energy transitions can only occur in quantized jumps of magnitude $l\hbar\omega$ follows as a natural consequence of the electrodynamic formulation. Radiative emission occurs when a l-microstate transits to a l'-microstate, with microstate energies given by (VII.10). Since boundary conditions *force* the index l to be an integer, the "quantum jump" phenomenon is thus explained. There are no fractional-order multipole moments to transit to, the smallest separation between multipole states being characterized by the requirement that Δl equal one.

It might be argued that the scalar wave-states associated with the quantum harmonic oscillator also possess this property, and therefore eclipse the need to investigate vector (electrodynamic) solutions to the same problem. But the electrodynamic formulation has a conceptual advantage in that no need arises to invoke artificial boundary conditions to get the problem to solve out properly. When solving the problem using harmonic oscillators, it is necessary to stipulate periodic boundary conditions on some fictional surface of dimension L^3 located far from the source. The electrodynamic formulation precludes this need. The electrodynamic boundary conditions conform to the actual contours of the radiation source.

CHAPTER VIII
APPLICATIONS

A.) Simple Example

The formulas of the previous sections represent the culmination of much mathematical labor, and once in hand, provide information about several aspects of the electromagnetic radiation field. Applying them to actual device structures is the best way to illustrate their utility. Three cases of practical interest are to be discussed in this final chapter of the report. In the initial section, a particularly simple geometry is used in order to best demonstrate the methodology.

Consider two halves of a thick spherical metal shell of radius R and infinite conductivity subjected to alternating potentials $\pm V \cos\omega t$. Determine the total power, force, and torque radiated from the sphere.

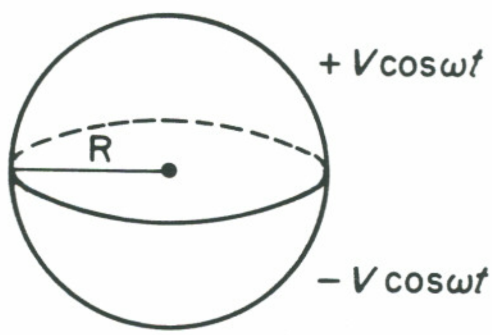


FIGURE 2

Spherically-shaped Antenna

Since the time-dependence is stipulated to be of monochromatic form, *i.e.*, $e^{-i\omega t}$, all the mathematical machinery of the previous two chapters is operable, in particular equation (III.F.7) for electromagnetic potential, equations (III.F.11 thru 16) for electromagnetic fields, and equations (VI.10 thru 16) for total radiated fluxes of energy, momentum, and angular momentum.

The expansion coefficients g_{lm} in the above formulas are determined from knowledge of $\psi|_{r=R}$.

The expansion coefficients d_{lm} in the above formulas are determined from knowledge of $\vec{E}\,|_{r=R}$.

We have been provided enough information to solve this problem in its entirety. Since $\vec{E}$ is zero inside the thick conductor, and since the tangential components of $\vec{E}$ must be continuous across the spherical boundary, one is forced to require that $e_\theta = e_\phi = 0$ at $r = R$.

Consequently, the following linear combination of e_θ and e_ϕ at $r = R$ must also equal zero:

$$0 = \left(e_\theta + ie_\phi\right)_{r=R} \tag{VIII.A.1}$$

$$= g_{lm}\left(\frac{dh_l^{(1)}(kr)}{dr} + \frac{h_l^{(1)}(kr)}{r}\right)_{r=R}\left(\frac{dP_l^m(\cos\theta)}{d\theta} - \frac{m}{\sin\theta}P_l^m(\cos\theta)\right)e^{im\phi}$$

$$+ \; k d_{lm}\, h_l^{(1)}(kR)\left(\frac{dP_l^m(\cos\theta)}{d\theta} - \frac{m}{\sin\theta}P_l^m(\cos\theta)\right)e^{im\phi}$$

Use equations (App.B2.2) and (App.B2.3) to obtain:

$$0 = k\Big(g_{lm}\Big(\frac{(l+1)h^{(1)}_{l-1}(kR) - l h^{(1)}_{l+1}(kR)}{(2l+1)}\Big) + d_{lm} h^{(1)}_{l}(kR)\Big) \cdot$$

$$\cdot \Big(\frac{dP^m_l(\cos\theta)}{d\theta} - \frac{m}{\sin\theta} P^m_l(\cos\theta)\Big) e^{im\phi}$$

Solve the above for d_{lm}:

$$d_{lm} = g_{lm}\Big(\frac{l h^{(1)}_{l+1}(kR) - (l+1)h^{(1)}_{l-1}(kR)}{(2l+1)h^{(1)}_{l}(kR)}\Big) \qquad \text{(VIII.A.2)}$$

Thus,

$$g_{lm}\, g^*_{lm} + d_{lm}\, d^*_{lm} = \qquad \text{(VIII.A.3)}$$

$$= g_{lm}\, g^*_{lm}\, \Big(1 + \frac{\big(l h^{(1)}_{l+1} - (l+1)h^{(1)}_{l-1}\big)\big(h^{(2)}_{l+1} - (l+1)h^{(2)}_{l-1}\big)}{(2l+1)h^{(1)}_{l}\, h^{(2)}_{l}}\Big)_{r=R}$$

where the argument $x = kR$ is understood for all $h^{(\alpha)}_{l}$ functions.

Examine the numerator of the second term within the large parantheses:

$$\left(lh^{(1)}_{l+1} - (l+1)h^{(1)}_{l-1}\right)\left(h^{(2)}_{l+1} - (l+1)h^{(2)}_{l-1}\right) \; = \qquad \text{(VIII.A.4)}$$

$$= \; l^2 h^{(1)}_{l+1} h^{(2)}_{l+1} \; - \; l(l+1)\left(h^{(1)}_{l+1} h^{(2)}_{l-1} + h^{(1)}_{l-1} h^{(2)}_{l+1}\right) \; + \; (l+1)^2 h^{(1)}_{l-1} h^{(2)}_{l-1}$$

Use equation (Appl.B2.15) on the center term:

$$= \; l^2 h^{(1)}_{l+1} h^{(2)}_{l+1} \; + \; l(l+1)\left(h^{(1)}_{l-1} h^{(2)}_{l-1} + h^{(1)}_{l+1} h^{(2)}_{l+1} - (2l+1)^2 \frac{1}{k^2 r^2} h^{(1)}_{l} h^{(2)}_{l}\right) \; +$$

$$+ \; (l+1)^2 h^{(1)}_{l-1} h^{(2)}_{l-1}$$

$$= \; (2l+1)\left[lh^{(1)}_{l+1} h^{(2)}_{l+1} \; + \; (l+1) h^{(1)}_{l-1} h^{(2)}_{l-1} \; - \; (2l+1)\frac{1}{k^2 r^2} h^{(1)}_{l} h^{(2)}_{l} \right]$$

Plug (VIII.A.4) back into (VIII.A.3) to obtain:

$$g_{lm}\, g^*_{lm} \; + \; d_{lm}\, d^*_{lm} \; = \qquad \text{(VIII.A.5)}$$

$$= \; g_{lm}\, g^*_{lm} \left(\frac{(l+1)h^{(1)}_{l-1} h^{(2)}_{l-1} \; + \; (2l+1) h^{(1)}_{l} h^{(1)}_{l} \; + \; lh^{(1)}_{l+1} h^{(1)}_{l+1}}{(2l+1) h^{(1)}_{l} h^{(2)}_{l}} \; - \right.$$

$$\left. - \; l(l+1)\frac{1}{k^2 r^2} \right)_{r=R}$$

Hence, for the frequent boundary condition that $e_\theta = e_\phi = 0$ at $r = R$, one has that:

$$W_U = \frac{c}{2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{l(l+1)}{(2l+1)} \frac{(l+m)!}{(l-m)!} g_{lm} g_{lm}^{*} \cdot \tag{VIII.A.6}$$

$$\cdot \left(\frac{(l+1) h_{l-1}^{(1)} h_{l-1}^{(2)} + (2l+1) h_{l}^{(1)} h_{l}^{(1)} + l h_{l+1}^{(1)} h_{l+1}^{(1)}}{(2l+1) h_{l}^{(1)} h_{l}^{(2)}} - l(l+1) \frac{1}{k^2 r^2} \right)_{r=R}$$

The above formula, although exact and perfectly well-suited for computer manipulation, is a bit complicated for analytical discussion. Therefore, in an attempt to simplify things for this section of the report, asymptotic limits of the above expression will be taken in the two opposing cases where kR is small and where kR is large. In both these instances, relatively simple asymptotic expressions for (VIII.A.6) can be obtained using (App.B3.8) and (App.B3.9).

For the limitting case that kR is small, one obtains:

$$\lim_{x \to \text{small}} \left(\frac{(l+1) h_{l-1}^{(1)}(x) h_{l-1}^{(2)}(x) + (2l+1) h_{l}^{(1)}(x) h_{l}^{(1)}(x) + l h_{l+1}^{(1)}(x) h_{l+1}^{(1)}(x)}{(2l+1) h_{l}^{(1)}(x) h_{l}^{(2)}(x)} - l(l+1) \frac{1}{x^2} \right) = \frac{l^2}{x^2} \tag{VIII.A.7}$$

Plugging the above into (VIII.A.6) yields the following asymptotic value for W_U:

$$W_U \xrightarrow[kR < 0.1]{} \frac{c}{2} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{l(l+1)}{(2l+1)} \frac{(l+m)!}{(l-m)!} \frac{l^2}{(kR)^2} g_{lm} g_{lm}^{*} \tag{VIII.A.8}$$

In the opposing case that kR is large, one has:

$$\lim_{x \to \text{large}} \left(\frac{(l+1)h_{l-1}^{(1)}(x)h_{l-1}^{(2)}(x) \ + \ (2l+1)h_l^{(1)}(x)h_l^{(1)}(x) \ + \ lh_{l+1}^{(1)}(x)h_{l+1}^{(1)}(x)}{(2l+1)h_l^{(1)}(x)h_l^{(2)}(x)} - l(l+1)\frac{1}{x^2} \right) = 2 \qquad \text{(VIII.A.9)}$$

In this situation, one has the asymptotic value for W_U:

$$W_U \xrightarrow[kR > 10]{} c \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{l(l+1)}{(2l+1)} \frac{(l+m)!}{(l-m)!} g_{lm} g_{lm}^* \qquad \text{(VIII.A.10)}$$

The expansion coefficients g_{lm} are determined from knowledge of ψ at $r = R$. In the geometry that we are dealing with here, this works out to be a simple calculation.

We have from (III.F.7) that:

$$\psi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} -g_{lm}(l+1)h_l^{(1)}(kr)P_l^m(\cos\theta)e^{im\phi}$$

Since the geometry being considered in this case displays polar symmetry, it can be inferred from the outset that ψ will contain no ϕ dependence. Hence, only the $m=0$ terms in the ψ-summation need be considered. The associated Legendre polynomials $P_l^m(\cos\theta)$ reduce down to the simple Legendre polynomials $P_l(\cos\theta)$ and one obtains:

$$\psi = \sum_{l=0}^{\infty} - g_{l0}(l+1)\, h_l^{(1)}(kr)\, P_l(\cos\theta) \qquad \text{(VIII.A.11)}$$

Because of the simple boundary condition at $r = R$, the ψ solution can be evaluated straightforwardly:

$$\psi|_{r=R} = \begin{cases} +V & \text{for } 0 \le \theta < \frac{\pi}{2} \\ -V & \text{for } \frac{\pi}{2} < \theta \le \pi \end{cases} \qquad \text{(VIII.A.12)}$$

Thus:

$$\sum_{l=0}^{\infty} - g_{l0}(l+1)\, h_l^{(1)}(kR)\, P_l(\cos\theta) = \begin{cases} +V & \text{for } 0 \le \theta < \frac{\pi}{2} \\ -V & \text{for } \frac{\pi}{2} < \theta \le \pi \end{cases} \qquad \text{(VIII.A.13)}$$

Multiply through by $P_{l'}(\cos\theta)\sin\theta\, d\theta$ and integrate over θ noting that:

$$\int_0^{\pi} P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta\, d\theta = \frac{2}{(2l+1)}\delta_{ll'} \qquad \text{(VIII.A.14)}$$

Therefore:

$$\sum_{l=0}^{\infty} \int_0^{\pi} - g_{l0}(l+1)\, h_l^{(1)}(kR)\, P_l(\cos\theta) P_{l'}(\cos\theta) \sin\theta\, d\theta = \qquad \text{(VIII.A.15)}$$

$$= \int_0^{\frac{\pi}{2}} (+V) P_{l'}(\cos\theta) \sin\theta\, d\theta + \int_{\frac{\pi}{2}}^{\pi} (-V) P_{l'}(\cos\theta) \sin\theta\, d\theta$$

Utilizing the orthogonality relation (VIII.A.14) on the L.H.S. and (App.C4.4 and 5) on the R.H.S., one obtains:

(VIII.A.16)

$$-g_{l0}(l+1)\,h_l^{(1)}(kR)\,\frac{2}{(2l+1)} = V\int_0^{\frac{\pi}{2}} P_l(\cos\theta)\sin\theta\,d\theta - V\int_{\frac{\pi}{2}}^{\pi} P_l(\cos\theta)\sin\theta\,d\theta$$

$$= V\left(P_l^{-1}(\tfrac{\pi}{2})\right) - V\left(-P_l^{-1}(\tfrac{\pi}{2})\right)$$

$$= 2V\left(P_l^{-1}(\tfrac{\pi}{2})\right)$$

$$= 2V\begin{cases} 0 & \text{for } l=\text{even} \\ \left(-\frac{1}{4}\right)^{\frac{l-1}{2}} \dfrac{(l-1)!}{(l+1)\left[\left(\frac{l-1}{2}\right)!\right]^2} & \text{for } l=\text{odd} \end{cases}$$

After some algebraic manipulation, one obtains:

$$g_{l0} = \begin{cases} 0 & \text{for } l=\text{even} \\ -\left(-\frac{1}{4}\right)^{\frac{l-1}{2}} \dfrac{(2l+1)}{(l+1)^2}\,\dfrac{(l-1)!}{\left[\left(\frac{l-1}{2}\right)!\right]^2}\,\dfrac{V}{h_l^{(1)}(kR)} & \text{for } l=\text{odd} \end{cases} \qquad \text{(VIII.A.17)}$$

Plugging these g_{l0} values into the general expression for the total radiated power (VIII.A.6) yields:

(VIII.A.18)

$$W_U = \frac{c}{2}\sum_{\substack{l=odd\\only}}^{\infty}\frac{l(l+1)}{(2l+1)}\left(\frac{1}{4}\right)^{l-1}\frac{(2l+1)^2}{(l+1)^4}\frac{\left[(l-1)!\right]^2}{\left[\left(\frac{l-1}{2}\right)!\right]^4}\frac{V^2}{h_l^{(1)}(kR)\,h_l^{(2)}(kR)}\;\cdot$$

$$\cdot\left(\frac{(l+1)h_{l-1}^{(1)}h_{l-1}^{(2)} + (2l+1)h_l^{(1)}h_l^{(1)} + lh_{l+1}^{(1)}h_{l+1}^{(1)}}{(2l+1)h_l^{(1)}h_l^{(2)}} - l(l+1)\frac{1}{k^2r^2}\right)_{r=R}$$

$$= \frac{c}{2}\sum_{\substack{l=odd\\only}}^{\infty}\left(\frac{1}{4}\right)^{l-1}\frac{l(2l+1)}{(l+1)^3}\frac{\left[(l-1)!\right]^2}{\left[\left(\frac{l-1}{2}\right)!\right]^4}\frac{V^2}{h_l^{(1)}(kR)\,h_l^{(2)}(kR)}\;\cdot$$

$$\cdot\left(\frac{(l+1)h_{l-1}^{(1)}h_{l-1}^{(2)} + (2l+1)h_l^{(1)}h_l^{(1)} + lh_{l+1}^{(1)}h_{l+1}^{(1)}}{(2l+1)h_l^{(1)}h_l^{(2)}} - l(l+1)\frac{1}{k^2r^2}\right)_{r=R}$$

The above formula is an exact analytical expression for the total radiated power from the adjoined pair of oscillating hemispheres as depicted in Figure 2. But such an expression is too complicated to continue handling analytically. It is best handled by reducing it to its small-kR and large-kR asymptotic limits.

First consider the case that kR is small. In this case, one has the (VIII.A.8) relation at one's disposal. Plugging in the (VIII.A.17) values for g_{l0} yields:

$$W_U \xrightarrow[kR<0.1]{} \frac{c}{2}\sum_{\substack{l=odd\\only}}^{\infty}\frac{l(l+1)}{(2l+1)}\frac{l^2}{(kR)^2}\left(\frac{1}{4}\right)^{l-1}\frac{(2l+1)^2}{(l+1)^4}\frac{\left[(l-1)!\right]^2}{\left[\left(\frac{l-1}{2}\right)!\right]^4}\frac{V^2}{h_l^{(1)}(kR)\,h_l^{(2)}(kR)}$$

(VIII.A.19)

(VIII.A.19′)

$$W_U \xrightarrow[kR<0.1]{} \frac{c}{2}\sum_{\substack{l=odd\\ only}}^{\infty}\left(\frac{1}{4}\right)^{l-1}\frac{l^3(2l+1)}{(l+1)^3}\frac{\left[(l-1)!\right]^2}{\left[\left(\frac{l-1}{2}\right)!\right]^4}\frac{1}{(kR)^2}\frac{V^2}{h_l^{(1)}(kR)\,h_l^{(2)}(kR)}$$

Using the small-kR asymptotic value (App.B3.8) for the conjuncted pair of spherical Hankel functions in the denominator, one obtains:

$$= \frac{c}{2}\sum_{\substack{l=odd\\ only}}^{\infty}\left(\frac{1}{4}\right)^{l-1}\frac{l^3(2l+1)}{(l+1)^3}\frac{\left[(l-1)!\right]^2}{\left[\left(\frac{l-1}{2}\right)!\right]^4}\frac{1}{(kR)^2}\frac{V^2}{\left[\frac{(2l-1)^2(2l-3)^2\cdots 3^2 1^2}{(kR)^{2l+2}}\right]}$$

$$= \frac{c}{2}\sum_{\substack{l=odd\\ only}}^{\infty}\left(\frac{1}{4}\right)^{l-1}\frac{l^3(2l+1)}{(l+1)^3}\frac{\left[(l-1)!\right]^2}{\left[\left(\frac{l-1}{2}\right)!\right]^4}\frac{V^2(kR)^{2l}}{\left[(2l-1)^2(2l-3)^2\cdots 3^2 1^2\right]}$$

Since kR is taken to be *small*, only the lowest-order term in the above series need concern us. Hence:

$$\lim_{kR\to\text{small}} W_U = \frac{c}{2}\frac{(1)^3(3)}{(2)^3}\frac{\left[0!\right]^2}{\left[0!\right]^4}\frac{V^2(kR)^2}{(1)^2} \qquad \text{(VIII.A.20)}$$

$$= \frac{3c}{16}V^2(kR)^2$$

$$= \frac{3}{16}\frac{V^2\omega^2R^2}{c}$$

Next, consider the case that kR is large. In this case, one has the (VIII.A.10) relation at one's disposal. Plugging the (VIII.A.17) values for g_{l0} into (VIII.A.10) yields:

$$W_U \xrightarrow[kR>10]{} c \sum_{\substack{l=odd \\ only}}^{\infty} \frac{l(l+1)}{(2l+1)} \left(\frac{1}{4}\right)^{l-1} \frac{(2l+1)^2}{(l+1)^4} \frac{\left[(l-1)!\right]^2}{\left[\left(\frac{l-1}{2}\right)!\right]^4} \frac{V^2}{h_l^{(1)}(kR)\, h_l^{(2)}(kR)}$$

$$= c \sum_{\substack{l=odd \\ only}}^{\infty} \left(\frac{1}{4}\right)^{l-1} \frac{l\,(2l+1)}{(l+1)^3} \frac{\left[(l-1)!\right]^2}{\left[\left(\frac{l-1}{2}\right)!\right]^4} \frac{V^2}{h_l^{(1)}(kR)\, h_l^{(2)}(kR)} \qquad \text{(VIII.A.21)}$$

Using the large-kR asymptotic value (App.B3.9) for the conjuncted pair of spherical Hankel functions in the denominator, one obtains:

$$= c \sum_{\substack{l=odd \\ only}}^{\infty} \left(\frac{1}{4}\right)^{l-1} \frac{l\,(2l+1)}{(l+1)^3} \frac{\left[(l-1)!\right]^2}{\left[\left(\frac{l-1}{2}\right)!\right]^4} \frac{V^2}{\left(\frac{1}{kR}\right)^2}$$

$$= c\, V^2 k^2 R^2 \sum_{\substack{l=odd \\ only}}^{\infty} \left(\frac{1}{4}\right)^{l-1} \frac{l\,(2l+1)}{(l+1)^3} \frac{\left[(l-1)!\right]^2}{\left[\left(\frac{l-1}{2}\right)!\right]^4}$$

Hence, for large kR, one obtains:

$$\lim_{kR \to \text{large}} W_U \;=\; \frac{V^2\omega^2R^2}{c} \sum_{\substack{l=odd\\ only}}^{\infty} \left(\frac{1}{4}\right)^{l-1} \frac{l\,(2l+1)}{(l+1)^3} \frac{\left[(l-1)!\right]^2}{\left[\left(\frac{l-1}{2}\right)!\right]^4}$$

$$= \frac{V^2\omega^2R^2}{c} \left[\text{Convergent Series in } l\right] \qquad \text{(VIII.A.22)}$$

Note that in both asymptotic limits, the power transfer characteristic goes as ω^2, implying that g_{lm} goes as ω.

Also, it is important to note that the above behavior is the radiative equivalent of a high-pass filter. The radiating source readily transmits high-ω signals, but suppresses low-ω signals, the transfer characteristic being quadratic in ω.

If a radiative transfer characteristic that is *not* high-pass is desired, it is obvious that (g_{lm}, d_{lm}) should *not* display a linear dependence on ω, but rather, some other more desirable dependence. This aspect will be explored in the following section of this report. But for now, it is necessary to complete this section by examining the values for radiated force and torque.

Since the even-l and non-zero-m g_{lm} and d_{lm} expansion coefficients are identically zero for this particular structure, the time-averaged momentum and angular momentum flux expressions (VI.7 thru 16) vanish. Hence, for this particular high-symmetry geometry, *i.e.*, one that manifests odd symmetry in θ and no dependence at all on ϕ, no net force or angular momentum is radiated. The only medium through which this object makes itself known to the external world is through its transfer of power. It can therefore increase the heat content of its environment or induce a transition in some photon-detecting device, but no external "winds" or "whirpools" are going to be generated.

Because no net force or torque is radiated away, it is clear that the given boundary conditions for this object disallow electromotively-induced rotation, translation, or pulsation of it. Less symmetric boundary conditions would relax these constraints, but also make the problem more difficult to solve analytically.

B.) Radiative Bandpass Structure

In Section A, a particular electromagnetic structure was given, for which it was necessary to determine multipole moments g_{lm} and d_{lm}. The converse problem, where multipole moments g_{lm} and d_{lm} are assumed given, but determination of the inferred electromagnetic structure is required, is the main objecive of this section. Besides being an interesting problem in its own right, it represents a novel application of the formulas of Chapters III and VI.

For the adjoined hemispheres of Section A, it was ultimately determined that expansion coefficients g_{lm} and d_{lm} terms were linear in ω. But this ω-dependence may not be desirable for many applications. The challenge before us is to utilize expansion coefficients with some desired property, and then work from there to determine the boundary surface S that generates these coefficients electromagnetically.

For example, consider the "clever" choice of g_{lm}:

$$g_{lm} = \begin{cases} \left(\frac{2l+1}{l(l+1)}\right)^{1/2}\left(\frac{\sin(\omega-\omega_o)\tau}{(\omega-\omega_o)\tau}\right)^{l} & \text{for } m=0 \\ 0 & \text{for all other } m \end{cases} \qquad \text{(VIII.B.1)}$$

Recall from (VI.10) that the energy flux is given as:

$$W_U = \frac{c}{2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l(l+1)}{(2l+1)}\frac{(l+m)!}{(l-m)!}\left(g_{lm}\,g_{lm}^{*} + d_{lm}\,d_{lm}^{*}\right)$$

which in our case becomes:

$$W_U = \frac{c}{2}\Big(1 + \Big(\frac{\sin(\omega-\omega_o)\tau}{(\omega-\omega_o)\tau}\Big)^2 + \Big(\frac{\sin(\omega-\omega_o)\tau}{(\omega-\omega_o)\tau}\Big)^4 + \Big(\frac{\sin(\omega-\omega_o)\tau}{(\omega-\omega_o)\tau}\Big)^6 + \cdots\Big]$$

$$= \frac{c}{2}\left[\frac{1}{1 - \Big(\frac{\sin(\omega-\omega_o)\tau}{(\omega-\omega_o)\tau}\Big)^2}\right] \quad \text{(VIII.B.2)}$$

$$= \text{resonant at } \omega_o$$

This should be contrasted with the results of the previous section, where it was determined that W_U was quadratic in ω; the hemispheres thus behaving as a high-pass filter for the emitted radiation. In contrast, the W_U of (VIII.B.2) is a strongly peaked function of ω at ω_o, thus behaving as a radiative bandpass filter centered at ω_o. The device designer has freedom to select ω_o for optimized emission at 1.55μm or 10.6μm, for instance. Essentially, a monochromatic diffraction grating has been concocted. Clearly, the merits of such a choice for g_{lm} are obvious; the problem arises as to how to achieve such a set of g_{lm} values.

This is where the ψ expression of (III.F.7) becomes so valuable:

$$\psi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} -g_{lm}(l+1)\, h_l^{(1)}(kr)\, P_l^m(\cos\theta)\, e^{im\phi}$$

Since all the g_{lm}'s are known, ψ is completely specified for all regions of space. Surfaces of *constant* ψ, *i.e.*, equipotential surfaces, represent the candidate topologies for boundary surface S.

The tactic to employ here is to generate surfaces of constant potential ψ using equations (III.F.7) and (VIII.B.1) as guides, or alternatively to calculate the three components of electric field (III.F.11 thru 13) at any given point to determine the vector normals of the equipotential surfaces. In actual practice, the second of the above techniques will most likely be the easier one to implement. Since ψ and $\vec{e}$ are such complicated functions of (r,θ,ϕ) and ω_o, it is clear that computer aided calculations are going to be necessary. This portion of the project is not done here, but instead reserved for a separate report later.

Some qualitiative features can be discussed however. For the proposed g_{lm}'s of (VIII.B.1), the parameter τ acts as an inverse measure of frequency bandwidth. Large τ corresponds to a sharply-peaked, narrow-range bandpass filter. Small τ corresponds to a smaller-peaked, broader-range filter. In actual applications, some intermediate τ value will probably be called for because "tuning" an extremely narrow-band grating will be difficult on the micron-scale. Further, there is the danger of mistuning the grating such that it chops out the desired signal portion of the spectrum. Thus, perfect tuning is probably not altogether desirable. Intermediate τ will perhaps be best, even though it entails pass-banding portions of the frequency spectrum closely adjoining the 1.55μm or 10.6μm carrier. Nevertheless, the vast majority of unuseful portions of the frequency spectrum do get properly suppressed. An enhanced signal-to-noise ratio should be the beneficial consequence.

Because of the sinusoid nature of the proposed g_{lm} coefficients, it can be safely pre-assumed that the boundary surface S is going to be corrugated, in keeping with diffraction grating structures in general. The equipotential surfaces traced out by (III.F.7) will be quite intricate, but due to Gauss's Divergence Theorem, all the formulas of Chapter VI will still hold, provided that the "deformed" surface S is topologically equivalent to a simply-connected sphere, and that the origin of the coordinate system used to define the spherical Hankel functions lies *inside* the simply-connected surface. Thus, all the mathematical machinery of Chapter VI remains completely valid, even for surfaces that at first glance would not appear amenable to exact analysis in spherical coordinates. In particular, parallelopieds and cyliners would qualify as suitable structures since they are simply-connected. Corrugated versions of these structures would also qualify. Tori and other such non-simply connected surfaces would *not* qualify, however. Neither would infinite planes or any 2-D surface that does not close back on itself.

The particular choice of sinc(x) functions in (VIII.B.1) was strictly for convenience and for purposes of illustrating the basic approach. In fact, any set of functions that display a strong local maximum at some given ω_o would have served just as well. Examples would be Gaussian functions, complemetary error functions, $(1 + (\omega-\omega_o)^2))^{-1}$ type functions, or any combination of such functions. The philosophy here is to judiciously select an infinite *set* of such functions, shifted such that their local maxima are all centered on the desired ω_o, and multiplied with an appropriate amplitude factor in order to properly exploit the energy flux formula of (VI.10). (In the example of VIII.B.1, this amplitude factor was selected as $((2l+1)/l(l+1))^{½}$). These terms can then be used as g_{lm} coefficients in the multipole expansion of the electromagnetic field; provided, of course, that they are not functions of (t,r,θ,ϕ), which would incur a violation of Maxwell's equations.

What has been accomplished here is the simultaneous utilization of the entire set of multipole moments to achieve a particular end. This is in contrast to the more typical case where one restricts attention to only the lowest-order term in an attempt to achieve the same end. It behooves one to utilize the full gamut of available resources when optimizing some desired performance characteristic. The above methods do this, and thus provide for greater power and flexibility in device design.

Refinements to these "crude stroke" ideas are clearly possible and will be pursued in a separate report.

C.) Collimated Beams

This section discusses the phenomenon of collimated electromagnetic beams. Such phenomena are clearly important. They are routinely observed under a variety of test conditions both inside and outside the laboratory, especially in the optical regime, where the entire subject of geometrical optics is based on the phenomenon. A correct description of such entities seems to be warranted.

It would at first appear that the Cartesian or cylindrical systems would serve as the correct framework with which to describe the phenomenon because attenuation along the direction of propagation, *viz.*, the z-axis, is postulated to be either non-existant or, in the case of evanescent modes, exponentially dependent upon z. The Cartesian and cylindrical coordinate systems single out this axis, thus any distinctive properties associated with it would be inherently easy to handle.

But it is precisely this under- or over-attenuation along the beam-axis that leads to infinite (or zero) energy and momentum fluxes far from the source, thus rendering the Maxwellian formalism of Chapter VI completely unusable. Since energy and momentum fluxes are quantities that typically need to be evaluated rather than discarded in problems of this sort, one is better served by working in a system that allows their usage. The spherical system is clearly what is called for as it automatically provides for $\vec{E}$ and $\vec{B}$ solutions that properly attenuate along the propagation axis of the beam. The objective is to devise spherical solutions that best approximate a true collimated beam, and utilize these spherical solutions in lieu of the overly-simplistic beams postulated earlier.

As will be made clear in the discussion ahead, the solution that is going to be presented in this section is, in a thermodynamic sense, the exact opposite to that examined in Chapter VII. In Chapter VII, an equipartition of energy was stipulated among all available radiation modes. In this section, the energy is postulated to reside exclusively in only one or two radiation modes, all other modes being suppressed. This situation represents the most extreme deviation from thermal equilibrium possible, and hence is induced under "extraordinary" circumstances, such as would be the case at the emission port of a laser cavity or in the cryogenically-cooled interior of a superconductor. In both these quoted examples, one or two quantum states are somehow induced to become populated to macroscopic levels, with an attendant display of quantum properties that would not be accessible under classical circumstances.

Since the topic discussed in this section is collimated beams, the electromagnetic emission from the laser cavity would be the more appropriate example to focus upon.

To begin the discussion, the time-independent vector amplitudes $\vec{e}$ and $\vec{b}$ as derived in equations (III.F.1 thru 6) are re-examined:

$$e_r = g_{lm}\, l(l+1)\frac{h_l^{(1)}(kr)}{r} P_l^m(\cos\theta)\, e^{im\phi} \tag{VIII.C.1}$$

$$e_\theta = g_{lm}\left(\frac{dh_l^{(1)}(kr)}{dr} + \frac{h_l^{(1)}(kr)}{r}\right)\frac{dP_l^m(\cos\theta)}{d\theta} e^{im\phi} \tag{VIII.C.2}$$

$$- \; k d_{l'm'}\, h_{l'}^{(1)}(kr)\frac{m'}{\sin\theta} P_{l'}^{m'}(\cos\theta)\, e^{im'\phi}$$

$$e_\phi = i g_{lm}\left(\frac{dh_l^{(1)}(kr)}{dr} + \frac{h_l^{(1)}(kr)}{r}\right)\frac{m}{\sin\theta} P_l^m(\cos\theta)\, e^{im\phi} \tag{VIII.C.3}$$

$$- \; ik d_{l'm'}\, h_{l'}^{(1)}(kr)\frac{dP_{l'}^{m'}(\cos\theta)}{d\theta} e^{im'\phi}$$

$$b_r = d_{l'm'}\, l'(l'+1)\frac{h_{l'}^{(1)}(kr)}{r} P_{l'}^{m'}(\cos\theta)\, e^{im'\phi} \tag{VIII.C.4}$$

$$b_r = d_{l'm'}\left(\frac{dh_{l'}^{(1)}(kr)}{dr} + \frac{h_{l'}^{(1)}(kr)}{r}\right)\frac{dP_{l'}^{m'}(\cos\theta)}{d\theta} e^{im'\phi} \tag{VIII.C.5}$$

$$+ \; k g_{lm}\, h_l^{(1)}(kr)\frac{m}{\sin\theta} P_l^m(\cos\theta)\, e^{im\phi}$$

$$b_\phi = i d_{l'm'}\left(\frac{dh_{l'}^{(1)}(kr)}{dr} + \frac{h_{l'}^{(1)}(kr)}{r}\right)\frac{m'}{\sin\theta} P_{l'}^{m'}(\cos\theta)\, e^{im'\phi} \tag{VIII.C.6}$$

$$+ \; ik g_{lm}\, h_l^{(1)}(kr)\frac{dP_l^m(\cos\theta)}{d\theta} e^{im\phi}$$

In the situation being investigated, it is assumed from the start that the radiated energy is contained entirely in just *two* radiation modes, namely, the electric multipole moment lm and the magnetic multipole moment $l'm'$. This condensation of energy into only a small packet of available modes is the essential feature of laser action, and as such, forms the mathematical backdrop for the remainder of this section.

No summations over the indices l and m are to be taken in the expressions (VIII.C.1 thru 6) since the two modes indicated are the only ones that are operative. The goal here is two select the two modes such that the virtually all the energy is concentrated in a cone of very small angle about the z-axis. In spherical terms, this means that the Poynting vector $1/8\pi(\vec{e}\times\vec{b}^*)$ is constrained to assume finite values only in regions of space where θ is very close to zero. Furthermore, since the emitted laser light is typically linearly polarized, the $\vec{e}$ vector is itself constrained to assume finite values only in regions of space where ϕ is zero or 180°. (Refer to Figure 3 for visual representation of angles θ and ϕ.) Both these constraints can be satisfied by proper choice of lm- and $l'm'$-modes, but before making this selection, a discussion of the relative magnitudes of the various terms in the $\vec{e}$ and $\vec{b}$ expressions of (VIII.C.1 thru 6) will prove helpful.

For those frequencies where $kr >> 1$, the $h_l^{(1)}/r$ and $dh_l^{(1)}/dr$ terms in the e_i and b_j expressions are overwhelmed by the $kh_l^{(1)}$ terms and can thus be safely discarded. Useful approximate expressions for the external electric and magnetic fields in these regimes are thus obtained by simply eliminating these small-order terms. Note that if r is to be taken on the scale of centimeters, the requirement that $kr >> 1$ forces $k=\omega/c$ to be in the infrared or visible regime. Thus, the truncated expressions obtained above would present no critical drawback in laser applications, but would pose serious contradictions in microwave or lowfrequency applications.

Specifically, asymptotic expressions for $\vec{e}$ and $\vec{b}$ in the limit that $kr >> 1$ are given as:

$$e_r \rightarrow \quad \text{Zero} \tag{VIII.C.7}$$

$$e_\theta \rightarrow \quad - k d_{l'm'} h_{l'}^{(1)}(kr) \frac{m'}{\sin\theta} P_{l'}^{m'}(\cos\theta) e^{im'\phi} \tag{VIII.C.8}$$

$$e_\phi \rightarrow \quad - ik d_{l'm'} h_{l'}^{(1)}(kr) \frac{dP_{l'}^{m'}(\cos\theta)}{d\theta} e^{im'\phi} \tag{VIII.C.9}$$

$$b_r \rightarrow \quad \text{Zero} \tag{VIII.C.10}$$

$$b_r \rightarrow \quad + k g_{lm} h_l^{(1)}(kr) \frac{m}{\sin\theta} P_l^m(\cos\theta) e^{im\phi} \tag{VIII.C.11}$$

$$b_\phi \rightarrow \quad + ik g_{lm} h_l^{(1)}(kr) \frac{dP_l^m(\cos\theta)}{d\theta} e^{im\phi} \tag{VIII.C.12}$$

In this approximation, the time-independent Poynting vector would go as:

$$\vec{s} \;=\; \frac{1}{8\pi}(\vec{e}\times\vec{b}^{*}) \;+\; c.c. \tag{VIII.C.13}$$

$$= \frac{\hat{r}}{8\pi}(e_\theta b^*{}_\phi + e_\phi b^*{}_\theta) \;+$$

$$+\; \frac{\hat{\theta}}{8\pi}(e_\phi b^*{}_r + e_r b^*{}_\phi) \;+$$

$$+\; \frac{\hat{\phi}}{8\pi}(e_r b^*{}_\theta + e_\theta b^*{}_\phi) \;+\; c.c.$$

$$= \frac{\hat{r}}{8\pi}\Bigg(ik^2 d_{l'm'}\, g^*_{lm}\, h^{(1)}_{l'}\, h^{(2)}_{l} \frac{m}{\sin\theta} P^m_l \frac{dP^{m'}_{l'}}{d\theta} \;+$$

$$+\; ik^2 d_{l'm'}\, g^*_{lm}\, h^{(1)}_{l'}\, h^{(2)}_{l} \frac{m'}{\sin\theta} P^{m'}_{l'} \frac{dP^m_l}{d\theta} \Bigg) \;+\; c.c.$$

Since it was earlier stipulated that the emitted beam must be linearly polarized, the indices m and m' are forced to be 1 or -1. Any other values would lead to roseate patterns for the transverse electric and magnetic fields. For convenience' sake, both these indices shall be set equal to 1.

It should also be noted that boundary conditions typically force the azimuthal indices l and l' to be equal. (The physical properties of the radiating source do not permit the electric and magnetic multipoles to possess different azimuthal symmetries.) Since this is the case, they will be taken equal for the remainder of this discussion.

With all these stipulations in mind, the Poynting vector simplifies to:

$$\vec{s} \;=\; \frac{\hat{r}}{8\pi}\,2ik^2\left(d_{lm}\,g^*_{lm} \;-\; d^*_{lm}\,g_{lm}\right)h^{(1)}_l\,h^{(2)}_l\,\frac{m}{\sin\theta}\,P^m_l\,\frac{dP^m_l}{d\theta} \qquad \text{(VIII.C.14)}$$

$$\rightarrow\; \frac{\hat{r}}{8\pi}\,2ik^2\left(d_{lm}\,g^*_{lm} \;-\; d^*_{lm}\,g_{lm}\right)\frac{1}{k^2r^2}\,\frac{m}{\sin\theta}\,P^m_l\,\frac{dP^m_l}{d\theta}$$

$$\rightarrow\; \frac{\hat{r}}{4\pi r^2}\,i\left(d_{lm}\,g^*_{lm} \;-\; d^*_{lm}\,g_{lm}\right)\frac{m}{\sin\theta}\,P^m_l\,\frac{dP^m_l}{d\theta}$$

where the index m equals 1 in the above formulas.

The dominant term of the associated Legendre polynomial of order l and index $m=1$ displays a θ dependence given by $\sin\theta\,(\cos\theta)^{l-1}$. Neglecting for the moment the remaining lower-order terms in the $P^m_l(\cos\theta)$ expression, one has:

$$\frac{m}{\sin\theta}\,P^m_l\,\frac{dP^m_l}{d\theta} \;\rightarrow\; \frac{m}{\sin\theta}\left(\sin\theta\,(\cos\theta)^{l-1}\right)\left(\frac{d}{d\theta}\left(\sin\theta\,(\cos\theta)^{l-1}\right)\right) \qquad \text{(VIII.C.15)}$$

$$=\; m\,(\cos\theta)^{l-1}\left[(\cos\theta)^{l} \;-\; (l-1)\sin^2\theta\,(\cos\theta)^{l-2}\right]$$

$$=\; m\,(\cos\theta)^{2l-3}\left[\cos^2\theta \;-\; (l-1)\sin^2\theta\right]$$

$$=\; m\,(\cos\theta)^{2l-3}\left[1 \;-\; l\sin^2\theta\right]$$

Plugging this last expression into the Poynting vector expression of (VIII.C.14) yields:

$$\bar{\text{s}} \rightarrow \frac{\hat{\text{r}}}{4\pi r^2}\, i\left(d_{lm}\, g_{lm}^{*} - d_{lm}^{*}\, g_{lm}\right)(\cos\theta)^{2l-3}\left[1 - l\sin^2\theta\right] \qquad \text{(VIII.C.14}'\text{)}$$

Since it is desired that the above function display delta-function-like behavior in θ, it is clear that the index l must be of *very high* order, *viz.*, 1000 or more. (Refer to Figure 4 for graphical representation of this high-order behavior.) With this last bit of knowledge, the program of proper mode determination is complete; the pair of electric and magnetic multipole moments that best simulate a collimated beam are those that have index m equal to 1 (or -1) and l of extremely high value. The emission characteristic of such an lm-mode is highly collimated (near the source), linearly-polarized, yet asymptotically zero at infinity. The attenuation characteritic is of the desired $1/r^2$ form to assure finite-valued energy and momentum fluxes.

A practitioner in the field might be aghast at the prospect of having to deal with a 1000th-order radiation mode, but such apprehensions are actually groundless when it is remembered that 1000th-order spherical Hankel and associated Legendre functions need *not* be calculated in order to determine total radiated power, force, and torque from such an object. General formulas derived in Chapter VI provide exact values for these quantities that do not require detailed knowledge of the field configuration. Knowledge of the multipole expansion coefficients g_{lm} and d_{lm} as determined from boundary conditions are sufficient to to completely determine radiated fluxes. For instance, from (VI.10), one has for the time-averaged energy flux:

$$\frac{1}{c}\left\langle W_U \right\rangle \;=\; \frac{1}{2}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{l(l+1)}{(2l+1)}\frac{(l+m)!}{(l-m)!}\left(g_{lm}\,g_{lm}^{*} \;+\; d_{lm}\,d_{lm}^{*}\right)$$

In the above formula, the two summations extend over only one term, namely, m equal to 1 and l equal to 1000. Plug in the appropriate values for g_{lm} and d_{lm} to obtain an *exact* expression for the radiated energy flux. Perform similar simple calculations for the other radiated quantities to complete the physical description of the radiating object.

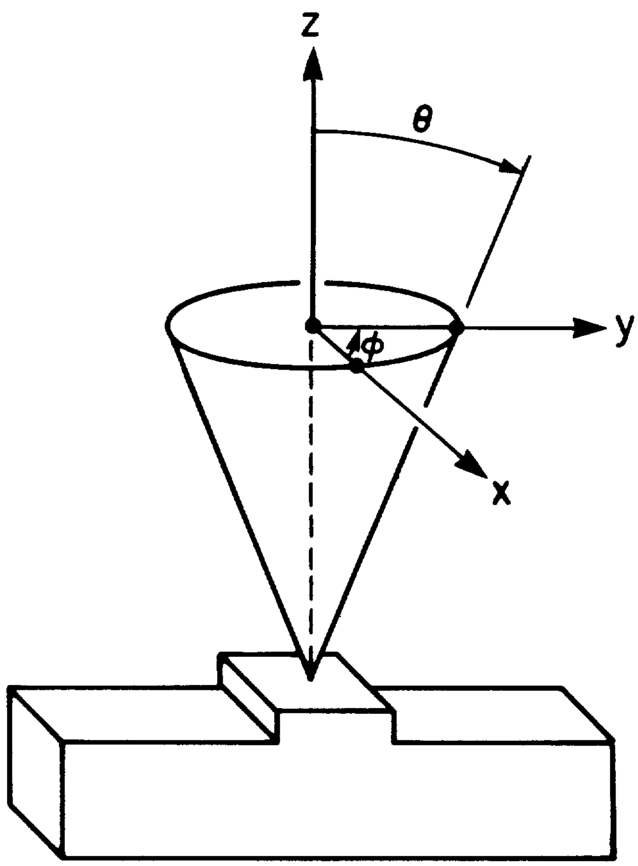


FIGURE 3

Configuration of Coordinate Axes in Discussion of Collimated Beams

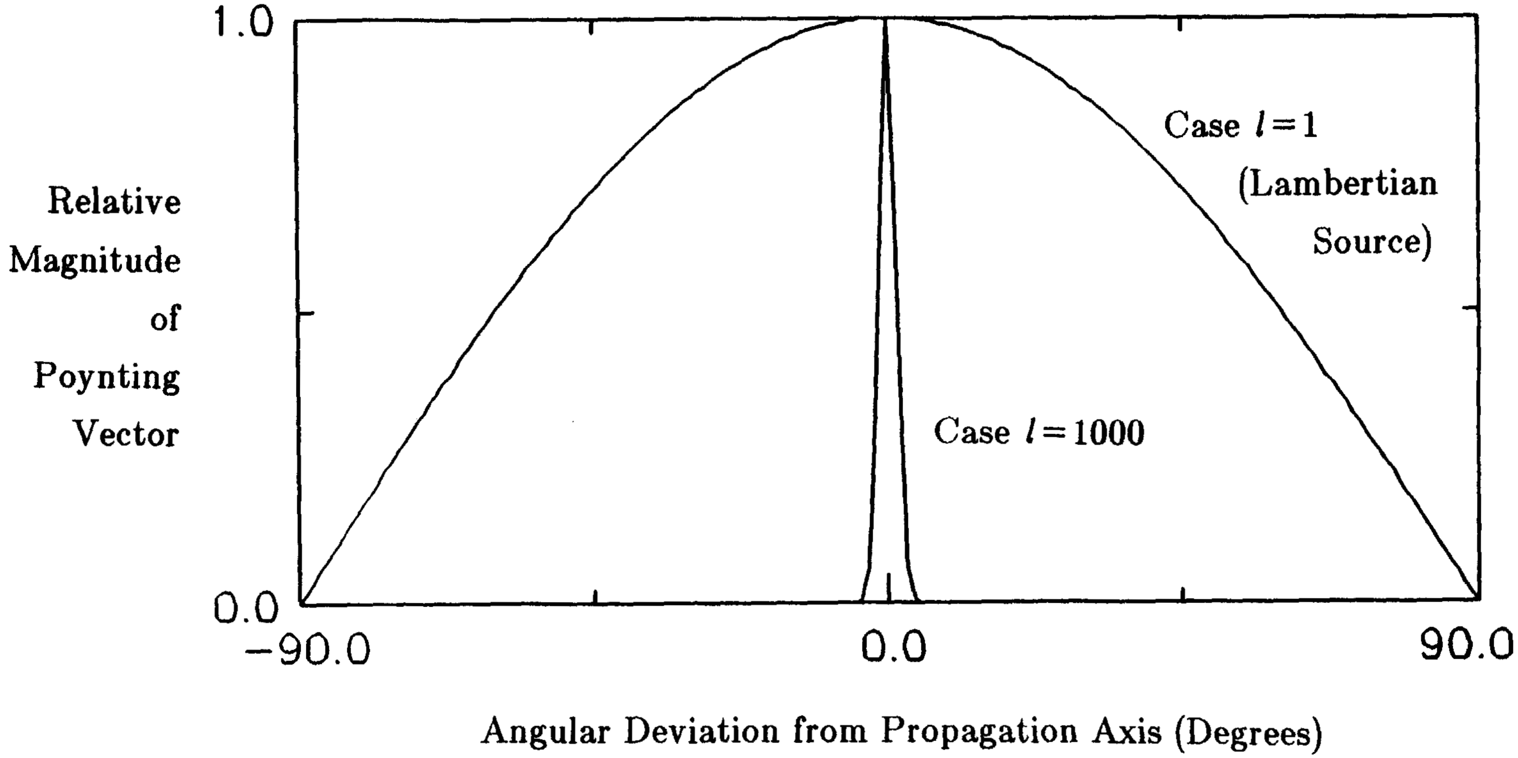


FIGURE 4

Azimuthal Characteristic of Poynting Vector for High-l and Low-l Order Beams

CHAPTER IX
CONCLUSION

The previous chapters have been devoted to the topic of electromagnetic fields in charge-free space. It has been shown that radiation fields emitted from stationary Gaussian surfaces find their natural expression in the spherical coordinate system. Treating partial differentiation as a tensor operation allows one to conveniently re-cast various differential identities such as divergence, gradient, and curl into the spherical system. Tensor-based arguments have thus been used to re-formulate Maxwell's equations into spherical form. A method of solving the homogeneous vector Helmholtz equation in the spherical system has been presented.

It has been noted that divergenceless functions of space and time possess many remarkable conservation properties. These conservation laws assumed especially cogent forms when expressed in the spherical system, and identifications of several electrodynamic quantities such as $1/8\pi(E^2+B^2)$ to quantities already familiar to us from fluid mechanics, such as power density, have been discussed. The material objects of mechanical physics (point masses, fluids, and so forth) are the tangible agents of energy and momentum transport; electromagnetic fields are the intangible agents of such transport.

It has also been demonstrated that a radiating Gaussian source is in a sense entirely characterized by its Maxwellian "DNA code", *i.e.*, its infinite set of electric and magnetic multipole moments. First-order combinations of these moments with their respective Maxwellian space-time function describe the electromagnetic properties of the external fields. Various quadratic combinations of these multipole moments describe the mechanical properties (power, force, and torque) radiated by these fields. This interplay between mechanical and electromagnetic properties indicates that "particle theories" (such as relativistic dynamics) are extractable from "wave theories" (Maxwellian electrodynamics) and vice versa. This duality between wave and particle theories has been a preoccupation of physicists since the time of Hamilton. This concept has not been exploited in the present investigation, but should certainly provide impetus to develop the theory further.

Another aspect explored in this investigation is the fruitful fusion of statistical mechanics and electrodynamics in explaining the emission spectra of radiating blackbodies. The electromagnetic formulas for energy and z-component of angular momentum are shown to be strongly reminiscent of their quantum mechanical analogs. Subjecting the individual energy components to the quantum hypothesis of Planck, and then developing the mathematics according to the dictates of statistical mechanics leads to a re-derivation of the blackbody emission spectrum. (The quantum of energy, $\hbar\omega$, must still enter the theory as an *ad hoc* hypothesis, however. The classical formulas do not yet supplant Schrödinger's or Heisenberg's first-principles approach.)

The interesting lesson gleaned from this exercise is that the classical (*i.e.*, Maxwellian) formulation contains the quantum solution as a special case. Once the electric and magnetic multipole moments are set equal to thermodynamically-determined functions of temperature and angular frequency, the blackbody emission formula falls out readily. This state of affairs upsets familiar notions about the hierarchy of physical theories. Typically, classical formulas are considered as a sort of sub-species of the more comprehensive quantum formulas. (Specifically, if $\hbar$ is allowed to approach zero in a quantum formula, the corresponding classical formula is recovered.) However, when classical formulas are supplemented with electromagnetic *fields*, the above hierarchy is likely to be reversed. It is possible that the classical theory forms the covering set for the corresponding quantum theory, and not the other way around. In order to explore this possibility to its fullest, it is necessary to expand the present formulation to include the case of non-zero $(\rho, \vec{J})$. This augmentation to regions of charged space should provide additional impetus to develop the theory further.

This is perhaps an opportune moment to mention a rough analogy between the E & M formulation and the two familiar versions of quantum mechanics, namely, the Schrödinger and the Heisenberg models. In the Schrödinger formulation, the physical content of any problem is entirely contained within a (scalar) wave function ψ that is itself a function of space and time. To obtain physical information from the ψ function, it must be acted on by some mathematical operator, then multiplied by its complex conjugate, and finally integrated over space to obtain a numerical value for the given mechanical quantity. Schrödinger's ψ function therefore acts as a half-density, that is, a function that must be multiplied by an altered version of itself before forming a true density that can then be integrated over space to provide the sought-for parametric value. This is directly analogous to Maxwell's $\vec{e}$ and $\vec{b}$ functions, which also must be combined quadratically before being integrated over space to obtain a numerical value for some given mechanical quantity.

Contrast this with the Heisenberg formulation where physical quantities are represented as infinitely-extended matrices with components p_{lm}. These matrices are then multiplied with other matrices to obtain relations such as $\tilde{x}\tilde{p} - \tilde{p}\tilde{x} = i\hbar$. Heisenberg's matrix elements p_{lm} correspond to Maxwell's g_{lm}

and d_{lm}, the electric and magnetic multipole moments. The energy, momentum, and angular momentum formulas of Chapter VI are somewhat akin to matrix-multiplications. The major difference to be aware of is that Heisenberg's matrices are square $N \times N$'s whereas the lm-multipole matrices are triangular, with $2l+1$ elements in each row l. Otherwise, the mathematical analogy holds fairly well.

Lastly, practical applications of the general formulas of earlier chapters are considered. Spherically-shaped antennas, surface diffraction gratings, and collimated beams are treated in the spherical formalism. Clearly, these treatments are rudimentary; more thorough-going analyses will be required to generate truly operational models.

REFERENCES

1. J. M. Blatt and V. Weisskopf, *Theoretical Nuclear Physics*, Dover, New York (1991)

2. E. H. Hill, "Theory of Vector Spherical Harmonics", Am. J. Physics, **22**, 211-214 (1954)

3. V. B. Berestetski, J. Expt. Theoret. Phys. (U.S.S.R.) **18**, 1057 (1948)

4. L. D. Landau and E. M. Lifshitz, *Course in Theoretical Physics*, *Vol. IV*, Pergamon, Tarrytown NY (1974)

5. C. J. Bouwkamp and H. B. G. Casimir, "On Multipole Expansions in the Theory of Electromagnetic Radiation", Physica, **20**, 539-554 (1954)

6. A. Nisbet, "Hertzian Electromagnetic Potentials and Associated Gauge Transformations", Proc. Roy. Soc., **231A**, 250-263 (1955)

7. J. A. Stratton, *Electromagnetic Theory*, McGraw-Hill, New York (1941)

8. W. K. Panofsky and M. Phillips, *Classical Electricity* and *Magnetism*, Addison-Wesley, Reading MA (1962)

9. J. D. Jackson, *Classical Electrodynamics*, John Wiley & Sons, New York (1975)

10. G. Arfken, *Mathematical Methods for Physicists*, Academic Press, San Diego (1985)

11. R. Becker, *Electromagnetic Fields and Interactions*, Dover, New York (1982)

12. D. Lovelock and H. Rund, *Tensors Differential Forms, and Variational Principles*, Dover, New York (1991)

13. T. Fulton, F. Rohrlich, and L. Witten, "Conformal Invariance in Physics", Rev. of Mod. Phys., **34**, 442-457 (1962)

14. F. Rohrlich, *Classical Charged Particles*, Addison-Wesley, Reading MA (1990)

15. W. Heitler, *The Quantum Theory of Radiation*, Dover, New York (1984)

16. O. Lummer and E. Pringsheim, *Verhandlungen der Deutschen physikalischen Gesellschaft*, **1**, 215 (1899); **2**, 163 (1900)

17. W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Functions of Mathematical Physics,* Chelsea Publishing, New York, 1949

APPENDIX

MATHEMATICAL FORMULAE

A.) Vector Differential Operators in Spherical Coordinates

$$\nabla\cdot\vec{V} = \frac{\partial V_r}{\partial r} + \frac{2}{r}V_r + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} + \frac{\cos\theta}{r\sin\theta}V_\theta + \frac{1}{r\sin\theta}\frac{\partial V_\phi}{\partial\phi} \qquad \text{(App.A.1)}$$

$$\nabla\psi = \hat{r}\frac{\partial\psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} \qquad \text{(App.A.2)}$$

$$\begin{aligned}\nabla\times\vec{V} = {} & \hat{r}\left(\frac{1}{r}\frac{\partial V_\phi}{\partial\theta} + \frac{\cos\theta}{r\sin\theta}V_\phi - \frac{1}{r\sin\theta}\frac{\partial V_\theta}{\partial\phi}\right) \\ & + \hat{\theta}\left(\frac{1}{r\sin\theta}\frac{\partial V_r}{\partial\phi} - \frac{\partial V_\phi}{\partial r} - \frac{1}{r}V_\phi\right) \qquad \text{(App.A.3)} \\ & + \hat{\phi}\left(\frac{\partial V_\theta}{\partial r} + \frac{1}{r}V_\theta - \frac{1}{r}\frac{\partial V_r}{\partial\theta}\right)\end{aligned}$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \qquad \text{(App.A.4)}$$

$$\begin{aligned}\nabla^2\vec{V} = {} & \hat{r}\left(\nabla^2 V_r - \frac{2}{r^2}V_r - \frac{2}{r^2}\frac{\partial V_\theta}{\partial\theta} - \frac{2\cos\theta}{r^2\sin\theta}V_\theta - \frac{2}{r^2\sin\theta}\frac{\partial V_\phi}{\partial\phi}\right) \\ & + \hat{\theta}\left(\nabla^2 V_\theta - \frac{1}{r^2\sin^2\theta}V_\theta + \frac{2}{r^2}\frac{\partial V_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial V_\phi}{\partial\phi}\right) \qquad \text{(App.A.5)} \\ & + \hat{\phi}\left(\nabla^2 V_\phi - \frac{1}{r^2\sin^2\theta}V_\phi + \frac{2}{r^2\sin^2\theta}\frac{\partial V_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial V_\theta}{\partial\phi}\right)\end{aligned}$$

Several expressions that prove useful in Chapter III are provided below.

Utilize (App.A.4) on the dot product $(\vec{r}\cdot\vec{V})$ and multiply the whole expression by $\vec{r}$:

$$\vec{r}\,\nabla^2(\vec{r}\cdot\vec{V}) = r\,\hat{r}\left[\nabla^2(r V_r)\right] \tag{App.A.6}$$

$$= \hat{r}\left(r^2\nabla^2 V_r + 2r\frac{\partial V_r}{\partial r} + 2V_r\right)$$

Multiply (App.A.1) by $\vec{r}$:

$$\vec{r}(\nabla\cdot\vec{V}) = \hat{r}\left(r\frac{\partial V_r}{\partial r} + 2V_r + \frac{\partial V_\theta}{\partial\theta} + \frac{\cos\theta}{\sin\theta}V_\theta + \frac{1}{\sin\theta}\frac{\partial V_\phi}{\partial\phi}\right) \tag{App.A.7}$$

Utilize (App.A.5) on the cross product $(\vec{r}\times\vec{V})$:

$$\nabla^2(\vec{r}\times\vec{V}) = \nabla^2\left[\hat{\theta}\,(-r V_\phi)+\hat{\phi}\,(r V_\phi)\right] \tag{App.A.8}$$

$$= \hat{r}\left(\frac{2}{r}\frac{\partial V_\phi}{\partial\theta} + \frac{2\cos\theta}{r\sin\theta}V_\phi + \frac{2}{r\sin\theta}\frac{\partial V_\theta}{\partial\phi}\right)$$

$$- \hat{\theta}\left(r\nabla^2 V_\phi + 2\frac{\partial V_\phi}{\partial r} + \frac{2}{r}V_\phi - \frac{1}{r\sin^2\theta}V_\phi + \frac{2\cos\theta}{r\sin^2\theta}\frac{\partial V_\theta}{\partial\phi}\right)$$

$$+ \hat{\phi}\left(r\nabla^2 V_\theta + 2\frac{\partial V_\theta}{\partial r} + \frac{2}{r}V_\theta - \frac{1}{r\sin^2\theta}V_\theta - \frac{2\cos\theta}{r\sin^2\theta}\frac{\partial V_\phi}{\partial\phi}\right)$$

Take the cross product of (App.A.8) with $\vec{r}$:

$$\vec{r}\times\nabla^2(\vec{r}\times\vec{V}) = -\hat{\theta}\left(r^2\nabla^2 V_\theta + 2r\frac{\partial V_\theta}{\partial r} + 2V_\theta - \frac{1}{\sin^2\theta}V_\theta - \frac{2\cos\theta}{\sin^2\theta}\frac{\partial V_\phi}{\partial\phi}\right)$$

$$- \hat{\phi}\left(r^2\nabla^2 V_\phi + 2r\frac{\partial V_\phi}{\partial r} + 2V_\phi - \frac{1}{\sin^2\theta}V_\phi + \frac{2\cos\theta}{\sin^2\theta}\frac{V_\theta}{\partial\phi}\right)$$

(App.A.9)

Take the cross product of (App.A.3) with $\vec{r}$:

$$\vec{r}\times(\nabla\times\vec{V}) = -\hat{\theta}\left(r\frac{\partial V_\theta}{\partial r} + V_\theta + \frac{\partial V_r}{\partial\theta}\right) + \hat{\phi}\left(\frac{1}{\sin\theta}\frac{\partial V_r}{\partial\phi} - r\frac{\partial V_\phi}{\partial r} - V_\phi\right)$$

(App.A.10)

Multiply equations (App.A.6), (App.A.7), (App.A.9), and (App.A.10) by $1/r^2$ and combine linearly to obtain:

$$\frac{\vec{r}}{r^2}\nabla^2(\vec{r}\cdot\vec{V}) - \frac{2\vec{r}}{r^2}(\nabla\cdot\vec{V}) + \frac{2\vec{r}}{r^2}\times(\nabla\times\vec{V}) - \frac{\vec{r}}{r^2}\times\nabla^2(\vec{r}\times\vec{V}) = \qquad \text{(App.A.11)}$$

$$= \hat{r}\left(\nabla^2 V_r - \frac{2}{r}V_r - \frac{2}{r^2}\frac{\partial V_\theta}{\partial\theta} - \frac{2\cos\theta}{r^2\sin\theta}V_\theta - \frac{2}{r^2\sin\theta}\frac{\partial V_\phi}{\partial\phi}\right)$$

$$+ \hat{\theta}\left(\nabla^2 V_\phi - \frac{1}{r^2\sin^2\theta}V_\theta + \frac{2}{r^2}\frac{\partial V_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial V_\phi}{\partial\phi}\right)$$

$$+ \hat{\phi}\left(\nabla^2 V_\phi - \frac{1}{r^2\sin^2\theta}V_\phi + \frac{2}{r^2\sin\theta}\frac{\partial V_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial V_\theta}{\partial\phi}\right)$$

$$= \nabla^2\vec{V}$$

B.) Spherical Hankel Functions

1.) Table of Lowest-Order Functions

Functions of First Kind:

(App.B1.1) $$h_0^{(1)}(x) = -i\,\frac{e^{ix}}{x}$$

(App.B1.2) $$h_1^{(1)}(x) = -\frac{e^{ix}}{x}\left(1+\frac{i}{x}\right)$$

(App.B1.3) $$h_2^{(1)}(x) = i\,\frac{e^{ix}}{x}\left(1+\frac{3i}{x}-\frac{3}{x^2}\right)$$

(App.B1.4) $$h_3^{(1)}(x) = \frac{e^{ix}}{x}\left(1+\frac{6i}{x}-\frac{15}{x^2}-\frac{15i}{x^3}\right)$$

(App.B1.5) $$h_4^{(1)}(x) = -i\,\frac{e^{ix}}{x}\left(1+\frac{10i}{x}-\frac{45}{x^2}-\frac{105i}{x^3}+\frac{105}{x^4}\right)$$

(App.B1.6) $$h_5^{(1)}(x) = -\frac{e^{ix}}{x}\left(1+\frac{15i}{x}-\frac{105}{x^2}-\frac{420i}{x^3}+\frac{945}{x^4}+\frac{945i}{x^5}\right)$$

In general:

(App.B1.7)

$$h_l^{(1)}(x) = (-i)^{l+1}\frac{e^{ix}}{x}\left[1 + \frac{l(l+1)}{2}\frac{i}{x} - \frac{(l-1)l(l+1)(l+2)}{2\cdot 4}\frac{1}{x^2} - \frac{(l-2)(l-1)l(l+1)(l+2)(l+3)}{2\cdot 4\cdot 6}\frac{i}{x^3} + \cdots\right]$$

Functions of Second Kind:

(App.B1.8) $$h_0^{(2)}(x) = i\,\frac{e^{-ix}}{x}$$

(App.B1.9) $$h_1^{(2)}(x) = -\frac{e^{-ix}}{x}\left(1-\frac{i}{x}\right)$$

(App.B1.10) $$h_2^{(2)}(x) = -i\,\frac{e^{-ix}}{x}\left(1-\frac{3i}{x}-\frac{3}{x^2}\right)$$

(App.B1.11) $$h_3^{(2)}(x) = \frac{e^{-ix}}{x}\left(1-\frac{6i}{x}-\frac{15}{x^2}+\frac{15i}{x^3}\right)$$

(App.B1.12) $$h_4^{(2)}(x) = i\,\frac{e^{-ix}}{x}\left(1-\frac{10i}{x}-\frac{45}{x^2}+\frac{105i}{x^3}+\frac{105}{x^4}\right)$$

(App.B1.13) $$h_5^{(2)}(x) = -\frac{e^{-ix}}{x}\left(1-\frac{15i}{x}-\frac{105}{x^2}-\frac{420i}{x^3}+\frac{945}{x^4}-\frac{945i}{x^5}\right)$$

In general:

(App.B1.14)

$$h_l^{(2)}(x) = (i)^{l+1}\frac{e^{-ix}}{x}\left[1 - \frac{l(l+1)}{2}\,\frac{i}{x} - \frac{(l-1)l(l+1)(l+2)}{2\cdot 4}\,\frac{1}{x^2} + \frac{(l-2)(l-1)l(l+1)(l+2)(l+3)}{2\cdot 4\cdot 6}\,\frac{i}{x^3} + \cdots\right]$$

B.) Spherical Hankel Functions

2.) Recursion Relations, Wronskian Relations, Other Identities

Throughout this section, paranthesized superscripts (α) and (β) are used. These superscripts may assume the values 1 or 2. Therefore, the expression $h_l^{(\alpha)}(kr)$ assumes one of two forms:

$$h_l^{(1)}(kr) = \text{Spherical Hankel Function of 1st Kind}$$

(App.B2.0) $$h_l^{(2)}(kr) = \text{Spherical Hankel Function of 2nd Kind} = \left(h_l^{(1)}(kr)\right)^*$$

(App.B2.1) $$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)h_l^{(\alpha)}(kr) = \left[\frac{l(l+1)}{r^2} - k^2\right]h_l^{(\alpha)}(kr)$$

(App.B2.2) $$\frac{dh_l^{(\alpha)}(kr)}{d(kr)} = \frac{1}{2l+1}\left[l h_{l-1}^{(\alpha)}(kr) - (l+1)h_{l+1}^{(\alpha)}(kr)\right]$$

(App.B2.3) $$\frac{h_l^{(\alpha)}(kr)}{kr} = \frac{1}{2l+1}\left[h_{l-1}^{(\alpha)}(kr) + h_{l+1}^{(\alpha)}(kr)\right]$$

(App.B2.4) $$h_l^{(1)}(kr)h_{l+1}^{(2)}(kr) - h_{l+1}^{(1)}(kr)h_l^{(2)}(kr) = \frac{2i}{k^2r^2}$$

(App.B2.5) $$h_{l-1}^{(1)}(kr)h_{l+1}^{(2)}(kr) - h_{l+1}^{(1)}(kr)h_{l-1}^{(2)}(kr) = (2l+1)\frac{2i}{k^3r^3}$$

(App.B2.6) $$\frac{dh_l^{(1)}(kr)}{dr}h_l^{(2)}(kr) - h_l^{(2)}(kr)\frac{dh_l^{(2)}(kr)}{dr} = \frac{2i}{kr^2}$$

For the relations that follow, the argument (kr) of the function $h_l^{(\alpha)}(kr)$ is understood. Hence, $h_l^{(\alpha)} \equiv h_l^{(\alpha)}(kr)$.

(App.B2.7) $$\left(\frac{d}{dr}+\frac{1}{r}\right)h_{l+1}^{(\alpha)} = kh_l^{(\alpha)} - (l+1)\frac{h_{l+1}^{(\alpha)}}{r}$$

(App.B2.8) $$\left(\frac{d}{dr}+\frac{1}{r}\right)h_l^{(\alpha)} = (l+1)\frac{h_l^{(\alpha)}}{r} - kh_{l+1}^{(\alpha)}$$

(App.B2.9) $$\left(\frac{d}{dr}+\frac{1}{r}\right)\left(\frac{h_{l+1}^{(\alpha)}}{r}\right) = \frac{k}{r}h_l^{(\alpha)} - \frac{(l+2)}{r^2}h_{l+1}^{(\alpha)}$$

(App.B2.10) $$\left(\frac{d}{dr}+\frac{1}{r}\right)\left(\frac{h_l^{(\alpha)}}{r}\right) = \frac{l}{r^2}h_l^{(\alpha)} - \frac{k}{r}h_{l+1}^{(\alpha)}$$

(App.B2.11) $$\left(\frac{d}{dr}+\frac{1}{r}\right)\left(\frac{dh_{l+1}^{(\alpha)}}{dr}\right) = \left(\frac{(l+2)^2}{r^2}-k^2\right)h_{l+1}^{(\alpha)} - \frac{k}{r}h_l^{(\alpha)}$$

(App.B2.12) $$\left(\frac{d}{dr}+\frac{1}{r}\right)\left(\frac{dh_l^{(\alpha)}}{dr}\right) = \left(\frac{l^2}{r^2}-k^2\right)h_l^{(\alpha)} + \frac{k}{r}h_{l+1}^{(\alpha)}$$

(App.B2.13) $$\frac{dh_{l+1}^{(\alpha)}}{dr} + (l+2)\frac{h_{l+1}^{(\alpha)}}{r} = kh_l^{(\alpha)}$$

(App.B2.14) $$\frac{dh_l^{(\alpha)}}{dr} - l\frac{h_l^{(\alpha)}}{r} = -kh_{l+1}^{(\alpha)}$$

(App.B2.15) $$(2l+1)^2\frac{h_l^{(\alpha)}}{kr}\frac{h_l^{(\beta)}}{kr} =$$
$$= h_{l-1}^{(\alpha)}h_{l-1}^{(\beta)} + h_{l-1}^{(\alpha)}h_{l+1}^{(\beta)} + h_{l+1}^{(\alpha)}h_{l-1}^{(\beta)} + h_{l+1}^{(\alpha)}h_{l+1}^{(\beta)}$$

B.) Spherical Hankel Functions

3.) Conjuncted Forms

The conjuncted pair of spherical Hankel functions $h_p^{(1)}(x)\,h_q^{(2)}(x)$ occurs so frequently in electromagnetic formulas that it warrants its own discussion. A new function $D_p^q(x)$ is defined as follows:

$$D_p^q(x) \equiv x^2 h_p^{(1)}(x)\,h_q^{(2)}(x) \qquad \text{(App.B3.1)}$$

Although not derived here, it can be shown that the $D_p^q(x)$ function satisfies its own fourth-order differential equation:

$$\begin{aligned} x^4\frac{d^4D_p^q}{dx^4} + 2x^3\frac{d^3D_p^q}{dx^3} + \left(4x^2 + 1 + (p-q)^2 + (p+q+1)^2\right)x^2\frac{d^2D_p^q}{dx^2} + \\ + \left(8x^2 - 1 + (p-q)^2 + (p+q+1)^2\right)x\frac{dD_p^q}{dx} + \\ + \left(1 - (p-q)^2 - (p+q+1)^2 + (p-q)^2(p+q+1)^2\right)D_p^q = 0 \end{aligned}$$

(App.B3.2)

From (App.B3.2), it is a quickly ascertained that:

$$D_p^q = a_1\Big[1 - \frac{\big((p-q)^2-1\big)\big((p+q+1)^2-1\big)}{4\cdot 2!}\frac{1}{x^2} + \frac{\big((p-q)^2-9\big)\big((p-q)^2-1\big)\big((p+q+1)^2-1\big)\big((p+q+1)^2-9\big)}{16\cdot 4!}\frac{1}{x^4} - \cdots\Big]$$

$$+ \; b_1\Big[\frac{1}{2\cdot 1!}\frac{1}{x} - \frac{\big((p-q)^2-4\big)\big((p+q+1)^2-4\big)}{8\cdot 3!}\frac{1}{x^3} + \frac{\big((p-q)^2-16\big)\big((p-q)^2-4\big)\big((p+q+1)^2-4\big)\big((p+q+1)^2-16\big)}{32\cdot 5!}\frac{1}{x^5} - \cdots\Big]$$

(App.B3.3)

After plugging in any pair of $h_p^{(1)}(x)$ and $h_q^{(2)}(x)$ functions into the above D_p^q expression, it is found that the amplitudes a_1 and b_1 are given as:

$$a_1 = (-i)^{(p-q)} \qquad \text{(App.B3.4)}$$

$$b_1 = (-i)^{(p-q)}\Big[i(p-q)(p+q+1)\Big] \qquad \text{(App.B3.5)}$$

For the special case that $p = q = l$, one obtains:

$$D_l^l(x) = 1 + \frac{1\cdot\left((2l+1)^2-1\right)}{4\cdot 2!}\frac{1}{x^2} + \frac{9\cdot 1\left((2l+1)^2-1\right)\left((2l+1)^2-9\right)}{16\cdot 4!}\frac{1}{x^4} + \cdots$$

(App.B3.6)

Some low-order values for the $D_l^l(x)$ function are:

(App.B3.7)

$$D_0^0(x) = 1 \quad \text{(a)}$$

$$D_1^1(x) = 1 + \frac{1}{x^2} \quad \text{(b)}$$

$$D_2^2(x) = 1 + \frac{3}{x^2} + \frac{9}{x^4} \quad \text{(c)}$$

$$D_3^3(x) = 1 + \frac{6}{x^2} + \frac{45}{x^4} + \frac{225}{x^6} \quad \text{(d)}$$

$$D_4^4(x) = 1 + \frac{10}{x^2} + \frac{135}{x^4} + \frac{1575}{x^6} + \frac{11025}{x^8} \quad \text{(e)}$$

In the two interesting limits of small x and large x, one obtains the asymptotic forms for $D_l^l(x) = x^2 h_l^{(1)}(x)\, h_l^{(2)}(x)$:

$$\lim_{x \to \text{small}} x^2 h_l^{(1)}(x)\, h_l^{(2)}(x) = \frac{(2l-1)^2(2l-3)^2 \cdots 3^2 1^2}{x^{2l}} \quad \text{(App.B3.8)}$$

$$\lim_{x \to \text{large}} x^2 h_l^{(1)}(x)\, h_l^{(2)}(x) = 1 \quad \text{(App.B3.9)}$$

C.) Associated Legendre Functions

1.) Table of Lowest-Order Functions

NOTE: The definition of Associated Legendre Polynomial $P_l^m(\cos\theta)$ for non-negative m in terms of the simple Legendre Polynomial $P_l(\cos\theta)$ as used in this report is given as:

$$P_l^m(\cos\theta) = (-1)^m (\sin\theta)^m \frac{d^m P_l(\cos\theta)}{(d\cos\theta)^m} \qquad \text{(Magnus-Oberhettinger phase)}$$

Beware that some authors omit the $(-1)^m$ factor in their definition of $P_l^m(\cos\theta)$, and consequently, minus signs that appear in the odd-parity expressions below will not appear in these authors' tables of the same functions.

This sign discrepancy also impacts recursion and integral formulas of upcoming sections. Any odd-parity $P_l^m(\cos\theta)$ term in formulas of subsequent sections must be preceded with a minus sign if the Magnus-Oberhettinger phase is *not* used. (Ref. 17)

$$P_0^0(\cos\theta) = 1 \tag{App.C1.1}$$

$$P_1^1(\cos\theta) = -\sin\theta \tag{App.C1.2}$$

$$P_1^0(\cos\theta) = \cos\theta \tag{App.C1.3}$$

$$P_1^{-1}(\cos\theta) = \frac{1}{2}\sin\theta \tag{App.C1.4}$$

$$P_2^2(\cos\theta) = 3\sin^2\theta \tag{App.C1.5}$$

$$P_2^1(\cos\theta) = -3\sin\theta\cos\theta \tag{App.C1.6}$$

$$P_2^0(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1) \tag{App.C1.7}$$

$$P_2^{-1}(\cos\theta) = \frac{1}{2}\sin\theta\cos\theta \tag{App.C1.8}$$

$$P_2^{-2}(\cos\theta) = \frac{1}{8}\sin^2\theta \tag{App.C1.9}$$

$$P_3^3(\cos\theta) = -15\sin^3\theta \quad \text{(App.C1.10)}$$

$$P_3^2(\cos\theta) = 15\sin^2\theta\cos\theta \quad \text{(App.C1.11)}$$

$$P_3^1(\cos\theta) = -\frac{3}{2}\sin\theta(5\cos^2\theta-1) \quad \text{(App.C1.12)}$$

$$P_3^0(\cos\theta) = \frac{1}{2}(5\cos^3\theta-3\cos\theta) \quad \text{(App.C1.13)}$$

$$P_3^{-1}(\cos\theta) = \frac{1}{8}\sin\theta(5\cos^2\theta-1) \quad \text{(App.C1.14)}$$

$$P_3^{-2}(\cos\theta) = \frac{1}{8}\sin^2\theta\cos\theta \quad \text{(App.C1.15)}$$

$$P_3^{-3}(\cos\theta) = \frac{1}{48}\sin^3\theta \quad \text{(App.C1.16)}$$

$$P_4^4(\cos\theta) = 105\sin^4\theta \quad \text{(App.C1.17)}$$

$$P_4^3(\cos\theta) = -105\sin^3\theta\cos\theta \quad \text{(App.C1.18)}$$

$$P_4^2(\cos\theta) = \frac{15}{2}\sin^2\theta(7\cos^2\theta-1) \quad \text{(App.C1.19)}$$

$$P_4^1(\cos\theta) = -\frac{5}{2}\sin\theta(7\cos^3\theta-3\cos\theta) \quad \text{(App.C1.20)}$$

$$P_4^0(\cos\theta) = \frac{1}{8}(35\cos^4\theta-30\cos^2\theta+3) \quad \text{(App.C1.21)}$$

$$P_4^{-1}(\cos\theta) = \frac{1}{4}\sin\theta(7\cos^3\theta-3\cos\theta) \quad \text{(App.C1.22)}$$

$$P_4^{-2}(\cos\theta) = \frac{1}{48}\sin^2\theta(7\cos^2\theta-1) \quad \text{(App.C1.23)}$$

$$P_4^{-3}(\cos\theta) = \frac{1}{48}\sin^3\theta\cos\theta \quad \text{(App.C1.24)}$$

$$P_4^{-4}(\cos\theta) = \frac{1}{384}\sin^4\theta \quad \text{(App.C1.25)}$$

C.) Associated Legendre Functions

2.) Recursion Relations, Other Identities

(App.C2.1) $$\left(\frac{d^2}{d\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{d}{d\theta}\right)P_l^m(\cos\theta) = \left[-l(l+1) + \frac{m^2}{\sin^2\theta}\right]P_l^m(\cos\theta)$$

(App.C2.2) $$P^m_{-l-1}(\cos\theta) = P_l^m(\cos\theta)$$

(App.C2.3) $$P_l^{-m}(\cos\theta) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(\cos\theta)$$

(App.C2.4) $$\cos\theta\, P_l^m(\cos\theta) = \frac{1}{2l+1}\left[(l-m+1)P_{l+1}^m(\cos\theta) + (l+m)P_{l-1}^m(\cos\theta)\right]$$

(App.C2.5) $$\sin\theta\, P_l^m(\cos\theta) = \frac{1}{2l+1}\left[P_{l-1}^{m+1}(\cos\theta) - P_{l+1}^{m+1}(\cos\theta)\right]$$

(App.C2.6) $$\sin\theta\, P_l^m(\cos\theta) = \frac{1}{2l+1}\Big[(l-m+1)(l-m+2)P_{l+1}^{m-1}(\cos\theta) -$$
$$- (l+m)(l+m-1)P_{l-1}^{m-1}(\cos\theta)\Big]$$

(App.C2.7) $$2m\frac{\cos\theta}{\sin\theta}P_l^m(\cos\theta) = -\left[P_l^{m+1}(\cos\theta) + (l+m)(l-m+1)P_l^{m-1}(\cos\theta)\right]$$

(App.C2.8) $$2m\frac{1}{\sin\theta}P_l^m(\cos\theta) = -\left[P_{l+1}^{m+1}(\cos\theta) + (l-m+1)(l-m+2)P_{l+1}^{m-1}(\cos\theta)\right]$$

(App.C2.9) $$2m\frac{1}{\sin\theta}P_l^m(\cos\theta) = -\left[P_{l-1}^{m+1}(\cos\theta) + (l+m)(l+m-1)P_{l-1}^{m-1}(\cos\theta)\right]$$

(App.C2.10) $$\frac{dP_l^m(\cos\theta)}{d\theta} = m\frac{\cos\theta}{\sin\theta}P_l^m(\cos\theta) + P_l^{m+1}(\cos\theta)$$

(App.C2.11) $$\frac{dP_l^m(\cos\theta)}{d\theta} = -m\frac{\cos\theta}{\sin\theta}P_l^m(\cos\theta) - (l+m)(l-m+1)P_l^{m-1}(\cos\theta)$$

(App.C2.12) $$\frac{dP_l^m(\cos\theta)}{d\theta} = \frac{1}{2}P_l^{m+1}(\cos\theta) - \frac{1}{2}(l+m)(l-m+1)P_l^{m-1}(\cos\theta)$$

For the relations that follow, the argument $(\cos\theta)$ of the function $P_l^m(\cos\theta)$ is understood. Hence, $P_l^m \equiv P_l^m(\cos\theta)$.

(App.C2.13)

$$\frac{dP_l^m}{d\theta} - \frac{m}{\sin\theta}P_l^m = \left(\frac{1-\cos\theta}{\sin^2\theta}\right)\frac{1}{2l+1}\left[lP_{l+1}^{m+1} + (2l+1)P_l^{m+1} + (l+1)P_{l-1}^{m+1}\right]$$

(App.C2.14)

$$\frac{dP_l^m}{d\theta} - \frac{m}{\sin\theta}P_l^m = \left(\frac{1+\cos\theta}{\sin^2\theta}\right)\frac{1}{2l+1}\Big[l(l-m+1)(l-m+2)P_{l+1}^{m-1}$$
$$- (2l+1)(l-m+1)(l+m)P_l^{m-1}$$
$$+ (l+1)(l+m-1)(l+m)P_{l-1}^{m-1}\Big]$$

(App.C2.15)

$$\frac{dP_l^m}{d\theta} + \frac{m}{\sin\theta}P_l^m = \left(\frac{1+\cos\theta}{\sin^2\theta}\right)\frac{1}{2l+1}\left[- lP_{l+1}^{m+1} + (2l+1)P_l^{m+1} - (l+1)P_{l-1}^{m+1}\right]$$

(App.C2.16)

$$\frac{dP_l^m}{d\theta} + \frac{m}{\sin\theta}P_l^m = \left(\frac{1-\cos\theta}{\sin^2\theta}\right)\frac{1}{2l+1}\Big[- l(l-m+1)(l-m+2)P_{l+1}^{m-1}$$
$$- (2l+1)(l-m+1)(l+m)P_l^{m-1}$$
$$- (l+1)(l+m-1)(l+m)P_{l-1}^{m-1}\Big]$$

C.) Associated Legendre Functions

3.) Definite Integrals

(App.C3.1) $$\int_0^{\pi} \frac{m}{\sin\theta} P_l^m P_l^{m'} \, d\theta = \frac{(l+m)!}{(l-m)!} \delta_{mm'} + (-1)^m \delta_{m(-m')}$$

(App.C3.2) $$\int_0^{\pi} P_l^m P_{l'}^m \sin\theta \, d\theta = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{ll'}$$

(App.C3.3) $$\int_0^{\pi} \left[\frac{m}{\sin\theta} \frac{dP_l^m}{d\theta} P_{l'}^m + \frac{m}{\sin\theta} P_l^m \frac{dP_{l'}^m}{d\theta} \right] \sin\theta \, d\theta = \text{Zero}$$

(App.C3.4) $$\int_0^{\pi} \left[\frac{dP_l^m}{d\theta} \frac{dP_{l'}^m}{d\theta} + \frac{m^2}{\sin^2\theta} P_l^m P_{l'}^m \right] \sin\theta \, d\theta = \frac{2l(l+1)}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{ll'}$$

C.) Associated Legendre Functions

4.) Indefinite Integrals

$$\left(l(l+1)-l'(l'+1)\right)\int_{\theta_1}^{\theta_2} P_l^m P_{l'}^{m'} \sin\theta \, d\theta \;-\; \left(m^2-m'^2\right)\int_{\theta_1}^{\theta_2} \frac{1}{\sin\theta} P_l^m P_{l'}^{m'} \, d\theta \;=$$

$$= \left[\sin\theta P_l^m \frac{dP_{l'}^{m'}}{d\theta} \;-\; \sin\theta \frac{dP_l^m}{d\theta} P_{l'}^{m'}\right]_{\theta_1}^{\theta_2} \qquad \text{(App.C4.1)}$$

A sub-case of (App.C4.1) that proves useful in Chapter VIII is the instance $l' = m' = 0$, for which one obtains:

$$l(l+1)\int_{\theta_1}^{\theta_2} P_l^m P_0^0 \sin\theta \, d\theta \;-\; m^2\int_{\theta_1}^{\theta_2} \frac{1}{\sin\theta} P_l^m P_0^0 \, d\theta \;=$$

$$= \left[\sin\theta P_l^m \frac{dP_0^0}{d\theta} \;-\; \sin\theta \frac{dP_l^m}{d\theta} P_0^0\right]_{\theta_1}^{\theta_2} \qquad \text{(App.C4.2)}$$

Utilize the explicit (App.C1.1) expression for P_0^0 to obtain:

$$= \left[-\sin\theta \frac{dP_l^m}{d\theta} \right]_{\theta_1}^{\theta_2}$$

Make use of (App.C2.11) to re-state the R.H.S. as:

$$= \left[m \cos\theta P_l^m \;+\; (l+m)(l-m+1)\sin\theta P_l^{m-1}\right]_{\theta_1}^{\theta_2}$$

Consider further that $m = 0$:

$$l(l+1)\int_{\theta_1}^{\theta_2} P_l \sin\theta \, d\theta = \left[l(l+1)\sin\theta P_l^{-1} \right]_{\theta_1}^{\theta_2}$$

Implying:

$$\int_{\theta_1}^{\theta_2} P_l \sin\theta \, d\theta = \left[\sin\theta P_l^{-1} \right]_{\theta_1}^{\theta_2} \quad \text{(App.C4.3)}$$

In particular:

$$\int_{0}^{\frac{\pi}{2}} P_l \sin\theta \, d\theta = \left[\sin\theta P_l^{-1} \right]_{0}^{\frac{\pi}{2}} = P_l^{-1}(\tfrac{\pi}{2}) - P_l^{-1}(0) = P_l^{-1}(\tfrac{\pi}{2}) \quad \text{(App.C4.4)}$$

And:

$$\int_{\frac{\pi}{2}}^{\pi} P_l \sin\theta \, d\theta = \left[\sin\theta P_l^{-1} \right]_{\frac{\pi}{2}}^{\pi} = P_l^{-1}(\pi) - P_l^{-1}(\tfrac{\pi}{2}) = - P_l^{-1}(\tfrac{\pi}{2}) \quad \text{(App.C4.5)}$$

D.) Complex Exponentials

1.) Definite Integrals

(App.D1.1)
$$\int_0^{2\pi} e^{im\phi} e^{-im'\phi} d\phi = 2\pi\delta_{mm'}$$

(App.D1.2)
$$\int_0^{2\pi} e^{im\phi} \cos\phi \, e^{-im'\phi} d\phi = \pi\left(\delta_{(m+1)m'} + \delta_{(m-1)m'}\right)$$

(App.D1.3)
$$\int_0^{2\pi} e^{im\phi} \sin\phi \, e^{-im'\phi} d\phi = -i\pi\left(\delta_{(m+1)m'} - \delta_{(m-1)m'}\right)$$

VITA

The author was born in Las Vegas, NV, on December 17, 1953. He received the B.S. degree in physics from the University of Redlands, California, in 1975, and the M.S.E. in Electrical Engineering from Princeton University, New Jersey, in 1976. His professional interests are in the field of semiconductor device physics. He was employed as Component Engineer for discrete semiconductor devices at Hughes Aircraft, Culver City, CA, for the year 1976-1977. He then worked at General Dynamics, Pomona, CA, 1977-1978, as Research Engineer on an InSb infrared detector program. This was followed in 1979-1980 by employment as Member of the Technical Staff at Santa Barbara Research Center, Goleta, CA, where he worked on HgCdTe infrared detector arrays. For the years 1981-1988, he worked at National Semiconductor, West Jordan, UT, as Senior Product Engineer on EPROM and SRAM devices. In 1988, he enrolled at the Oregon Graduate Institute, where his research projects involved work on ZnS-based electroluminescent devices, plasma processing, and MOCVD growth of AlGaAs/GaAs and InGaAs/InP structures.