A Coordinate-Independent Center Manifold Reduction

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Abstract

This note gives a method for performing the center manifold reduction that eliminates the need to transform the original equations of motion into block diagonal form. To achieve this, we write the center manifold as an embedding, rather than as a graph over the center subspace. The technique is well-suited to computer algebra implementation of the center manifold reduction.

1 Introduction

The center manifold reduction is a technique for eliminating non-essential degrees of freedom in bifurcation problems. The low-dimensional equations of motion on the center manifold, or their projection onto the center subspace, tell us about the topological character of the flow.

For example suppose that the vector field $f(X) : R^N \to R^N$ has an equilibrium at $X = 0$. Let Df_0 be the linearization at this equilibrium. The center subspace E^c is the space spanned by the (generalized) eigenvectors of Df_0 corresponding to eigenvalues on the imaginary axis. The center manifold is tangent to E^c at $X = 0$ and is invariant under the flow of f.

2 The Graph Construction

In the usual center manifold reduction, one transforms the equations of motion into eigencoordinates. One then writes the center manifold as a graph over the center subspace, the latter having been linearly decoupled from the other degrees of freedom by the diagonalization [1, 2].

Assume that the vector field has an equilibrium at the origin. In the transformed coordinates, the system of differential equations has the form [1, p. 130]),

$$\begin{aligned} \dot{x} &= Bx + F(x, y) \\ \dot{y} &= Cy + G(x, y) \end{aligned} \tag{1}$$

where B has eigenvalues with real part equal to zero, C has eigenvalues with negative real part, and F and G and their first derivatives vanish at the origin. We assume that the unstable manifold is empty. The center manifold is written as a graph

$$W^c = \{(x, y) | y = h(x)\}, \quad h(0) = Dh(0) = 0 \tag{2}$$

in the neighborhood of the origin. Since the center manifold is invariant under the flow, we can substitute $y = h(x)$ into the second equation of (1) and obtain

$$Dh(x)\,[\,Bx + F(x, h(x))\,] \;-\; Ch(x) \;-\; G(x, h(x)) \;=\; 0 \tag{3}$$

This is solved for $h(x)$ by expanding in a power series about the origin with the boundary conditions $h(0) = Dh(0) = 0$.

With the center manifold identified, the flow is projected onto the center subspace

$$\dot{x} \;=\; Bx + F(x, h(x)). \tag{4}$$

The stability of the origin for the full system (1) is given by the stability for the reduced system (4).

3 The Center Manifold as an Embedding

One difficulty in the above procedure is that the initial transformation to simplify the linear system can be quite cumbersome. For high-dimensional systems the diagonalization becomes tedious for hand calculation and one turns to a machine implementation. Unfortunately, diagonalizing large algebraic matrices can be difficult for symbolic computation systems.

This note gives an alternative center manifold expansion that dispenses with the need to perform the initial coordinate transformation. The procedure requires only knowledge of the vectors spanning the center subspace. For systems where Df_0 has zero eigenvalues, the center subspace is just the kernel of Df_0. Symbolic computation packages are able to find kernels of matrices with little difficulty.

We return to the original system of equations

$$\dot{x} = f(x) \tag{5}$$

and assume that $f(0) = 0$ and that Df_0 has a one-dimensional kernel spanned by v_τ. Before proceeding, we need to establish some notation. $Df[a]$ is the action of Df on the vector a. The result is a vector whose components in a coordinate system are

$$(Df[a])^i \equiv \sum_{j=1}^{N} \frac{\partial f^i}{\partial x^j} a^j.$$

where a^i is the i^{th} component of the vector a. For example

$$Df_0[v_\tau] = 0.$$

Similarly $D^2f[a,b]$ is the action of the second derivative D^2f on the pair of vectors a, b. The result is a vector with components

$$(D^2f[a,b])^i \equiv \sum_{j,k} \frac{\partial^2 f^i}{\partial x^j \partial x^k} a^j b^k.$$

Contractions of higher order derivatives are similarly defined.

We write the center manifold as an embedding from $R^1 \rightarrow R^N$,

$$x_{cm}(\tau) = v_\tau \tau + w(\tau) = v_\tau \tau + \frac{1}{2} w_{\tau\tau} \tau^2 + \ldots . \tag{6}$$

$$w(0) = 0, \quad \frac{dw(0)}{d\tau} \equiv w_\tau(0) = 0 \tag{7}$$

with $w \perp v_\tau$. (Here we choose w to lie in the orthogonal complement to v_τ. One can use another splitting, e.g. choose w in the range of Df_0. This leads to a different parameterization of the center manifold, i.e. $w(\tau)$ has a different functional form. This is illustrated in the example in §4.) The form of the embedding (6) and the boundary conditions (7) insure that the center manifold passes through the equilibrium at the origin, and that it is tangent to the center subspace E^c there. The geometry of the construction is shown in figure 1.

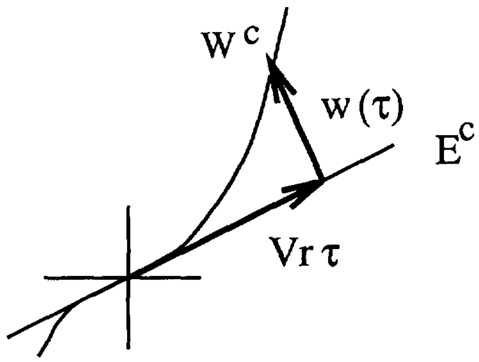

Figure 1: Center manifold W^c and center subspace E^c.

Since the center manifold is invariant under the flow, the vector field on x_{cm} must be tangent to x_{cm}. Hence we can write

$$
\begin{aligned}
f(x_{cm}(\tau)) &= \alpha(\tau)\, \frac{dx_{cm}}{d\tau} \\
&= \alpha(\tau)\, (\, v_r + w_\tau(\tau)\,) \qquad (8)
\end{aligned}
$$

where $\alpha(\tau)$ is a real-valued function. Note that since $f(0) = 0$, $\alpha(0) = 0$.

3.1 Flow on the Center Manifold

The flow on the center manifold is given by

$$
\dot{x}_{cm} \;=\; \frac{dx_{cm}}{d\tau}\, \dot{\tau} \;=\; f(\, x_{cm}(\tau)\,) \;=\; \frac{dx_{cm}}{d\tau}\, \alpha(\tau), \qquad (9)
$$

having used (8) in the last equality. Thus

$$
\dot{\tau} \;=\; \alpha(\tau),
$$

and the scaling function $\alpha(\tau)$ gives the time rate of change of the embedding parameter induced by the motion on the center manifold.

3.2 Solving the Embedding Equation

The embedding function w and the scaling function α are found by solving the tangency condition (8) in a Taylor series expansion about $\tau = 0$. For the first order term in the Taylor expansion, take the derivative of (8) with respect to τ

$$Df\,[\frac{dx_{cm}}{d\tau}] \;=\; Df\,[\,v_r + w_\tau\,] \;=\; \alpha_\tau\,(v_r + w_\tau) + \alpha\,w_{\tau\tau}. \tag{10}$$

Evaluating (10) at $\tau = 0$ gives

$$\alpha_{\tau_0} \equiv \alpha_\tau(0) \;=\; 0 \tag{11}$$

which expresses the fact that the linear part of the motion along the center manifold vanishes at the origin. For the second order term, take the derivative of (10) with respect to τ leaving

$$\begin{aligned} Df\,[\,w_{\tau\tau}\,] \;&+\; D^2 f\,[v_r + w_\tau,\; v_r + w_\tau\,] \\ &=\; \alpha_{\tau\tau}\,(v_r + w_\tau) \;+\; 2\,\alpha_\tau\,w_{\tau\tau} \;+\; \alpha\,w_{\tau\tau\tau}. \end{aligned}$$

Evaluating this at $\tau = 0$ leaves

$$Df_0\,[\,w_{\tau\tau}\,] \;+\; D^2 f_0\,[v_r, v_r] \;=\; \alpha_{\tau\tau_0}\,v_r \tag{12}$$

Now let v_l be the left eigenvector of Df_0 corresponding to eigenvalue zero. Left multiply the last expression by v_l and solve for $\alpha_{\tau\tau_0}$ to obtain

$$\alpha_{\tau\tau_0} \;=\; v_l \cdot D^2 f_0[v_r, v_r] \,/\, (v_l \cdot v_r). \tag{13}$$

With this expression for $\alpha_{\tau\tau_0}$, (12) can be solved for $w_{\tau\tau_0}$

$$w_{\tau\tau_0} \;=\; L^{-1}\,(\,\alpha_{\tau\tau_0}\,v_r \;-\; D^2 f_0\,[\,v_r, v_r\,]\,) \tag{14}$$

where L is defined as the restriction of Df_0 to $v_r^{\perp}$, and L^{-1} is its inverse. [1]

The procedure can be extended to arbitrary order. For reference the third order terms are

$$\alpha_{\tau\tau\tau_0} = \frac{v_l \cdot (\, 3\, D^2 f_0[v_r, w_{\tau\tau_0}] + D^3 f_0[v_r, v_r, v_r] - 3\alpha_{\tau\tau_0}(v_l \cdot w_{\tau\tau_0}))}{v_l \cdot v_r} \qquad (15)$$

$$w_{\tau\tau\tau_0} = L^{-1}(\alpha_{\tau\tau\tau_0}, v_r + 3\alpha_{\tau\tau_0} w_{\tau\tau_0} - 3D^2 f_0[w_{\tau\tau_0}, v_r] - D^3 f_0[v_r, v_r, v_r]). \qquad (16)$$

Note that to obtain α to third order (and hence the motion on the center manifold to third order), we need w only to second order.

4 An Example: The Lorenz Equations

The Lorenz equations

$$\dot{x} = \sigma\,(y - x) \qquad (17)$$

$$\dot{y} = \rho x - y - xz \qquad (18)$$

$$\dot{z} = xy - \beta z \qquad (19)$$

have an equilibrium at $(x, y, z) = (0, 0, 0)$ with linearization

$$Df_0 = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}. \qquad (20)$$

At $\rho = 1$, Df_0 has right and left kernels $v_r = \{1, 1, 0\}$, $v_l = \{1/\sigma, 1, 0\}$.

The center manifold is given as an embedding from (τ, ρ) into (x, y, z, ρ) with the tangency condition

$$f(\, x_{cm}(\tau, \rho), \rho\,) = \alpha\,(\tau, \rho)\,(\, v_r + w_\tau(\tau, \rho)\,). \qquad (21)$$

[1] Df_0 is an isomorphism from $v_r^{\perp}$ to *Range*(Df_0) and, from (12), $\alpha_{\tau\tau}\, v_r - D^2 f_0\,[v_r, v_r]$ is in *Range*(Df_0), so $w_{\tau\tau_0}$ is uniquely defined in $v_r^{\perp}$.

Differentiating this with respect to ρ leaves

$$Df\,[\,w_\rho\,] + f_\rho \;=\; \alpha_\rho\,(\,v_r + w_\tau\,) \;+\; \alpha\, w_{\tau\rho}, \tag{22}$$

where $f_\rho \equiv \partial f/\partial\rho$. Evaluating the derivatives at the bifurcation point $(\tau,\rho) = (0,1)$ and dotting with v_l gives

$$\alpha_{\rho_0} = 0.$$

Substituting this result into (22) and evaluating the result at the bifurcation point leaves

$$Df_0\,[\,w_{\rho_0}\,] = 0.$$

Since $w \perp v_r$, this requires that

$$w_{\rho_0} = 0.$$

Differentiating (22) with respect to τ gives

$$\begin{aligned} &D^2 f\,[\,w_\rho, v_r + w_\tau\,] \;+\; Df\,[\,w_{\rho\tau}\,] \;+\; Df_\rho\,[\,v_r + w_\tau\,] \\ &= \alpha_{\tau\rho}\,(\,v_r + w_\tau\,) \;+\; \alpha_\tau\, w_{\tau\rho} \;+\; \alpha_\rho\, w_{\tau\tau} \;+\; \alpha\, w_{\tau\tau\rho}. \end{aligned} \tag{23}$$

Evaluating this at the bifurcation point leaves

$$\begin{aligned} Df_0\,[\,w_{\tau\rho_0}\,] \;&+\; Df_{\rho_0}\,[\,v_r\,] \\ &=\; \alpha_{\tau\rho_0}\, v_r. \end{aligned}$$

The last expression is solved for $\alpha_{\tau\rho_0}$ and $w_{\tau\rho_0}$ as

$$\alpha_{\tau\rho_0} \;=\; \frac{v_l \cdot (\,Df_{\rho_0}\,[\,v_r\,])}{v_l \cdot v_r} \tag{24}$$

$$w_{\tau\rho_0} \;=\; L^{-1}\,(\,\alpha_{\tau\rho_0}\, v_r - Df_{\rho_0}\,[\,v_r\,]\,). \tag{25}$$

With

$$Df_{\rho_0} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

I find

$$\alpha_{\tau\rho_0} = \frac{\sigma}{1+\sigma} \tag{26}$$

$$w_{\tau\rho_0} = \left\{ \frac{-1}{2(1+\sigma)}, \frac{1}{2(1+\sigma)}, 0 \right\}. \tag{27}$$

The terms second order in τ are given by (13) and (14). The required contraction is

$$D^2 f_0 [\, v_r, v_r \,] = \{0, 0, 2\}$$

and I find

$$\alpha_{\tau\tau_0} = 0 \tag{28}$$

$$w_{\tau\tau_0} = \{0, 0, 2/\beta\}. \tag{29}$$

Thus to second order the center manifold is given by

$$\begin{aligned} x_{cm}(\tau, \rho) &= v_r\, \tau + w_{\tau\rho_0}\, \tau(\rho - 1) + \frac{1}{2} w_{\tau\tau_0}\, \tau^2 \\ &= \left\{ \frac{\tau\,(\,(1-\rho) + 2(1+\sigma)\,)}{2(1+\sigma)},\ \frac{\tau(\,1+\rho+2\sigma\,)}{2(1+\sigma)},\ \tau^2/\beta \right\}. \end{aligned} \tag{30}$$

To calculate the motion on the center manifold, we need the coefficient $\alpha_{\tau\tau\tau_0}$ from equation (15). Since $D^3 f$ and $\alpha_{\tau\tau_0}$ are both zero, we have

$$\alpha_{\tau\tau\tau_0} = \frac{3 v_l \cdot D^2 f_0 [v_r, w_{\tau\tau_0}]}{v_l \cdot v_r} = \frac{-6\sigma}{\beta(1+\sigma)}.$$

Including only terms of order τ^3 and $(\rho - 1)\,\tau$, the motion on the center manifold is given by

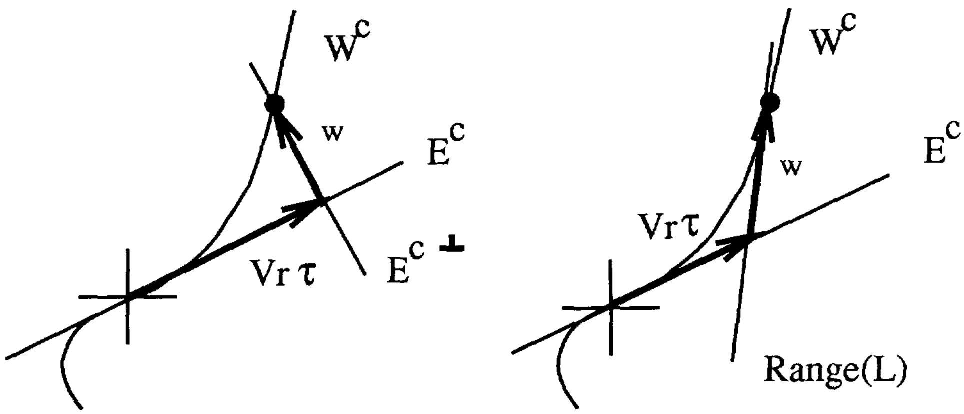

Figure 2: Two different parameterizations of the center manifold.

$$\begin{aligned} \dot{\tau} \;=\; \alpha(\tau) &= \frac{1}{3!}\,\alpha_{\tau\tau\tau_0}\,\tau^3 \;+\; \alpha_{\tau\rho_0}\,\tau\,(\rho-1) \;+\; \ldots \\ &= -\frac{\sigma}{\beta(1+\sigma)}\,\tau^3 \;+\; \frac{\sigma}{1+\sigma}\,\tau\,(\rho-1)\ldots \end{aligned} \tag{31}$$

which is the normal form for a pitchfork bifurcation.

4.1 Alternative Parameterization

In the construction used so far, we wrote the center manifold in terms of components in E^c and $E^{c\perp}$ as depicted in figure 2a. The constraint $w(\tau) \perp v_r$ uniquely defines the preimage of Df_0 used in (14) and (16). One can, however, use any other convenient decomposition. For example we can require $w(\tau) \in \text{Range}(Df_0)$ as in figure 2b. This defines a different preimage of Df_0 for use in (14) and (16) and will lead to different functional forms for $w(\tau)$ and $\alpha(\tau)$. These differences amount to a reparameterization of the center manifold.

For the example of the Lorenz system, writing the center manifold as

$$x_{cm} \;=\; v_r\,\tilde{\tau} \;+\; w(\tilde{\tau},\rho), \quad w \in \text{Range}(Df_0)$$

leads to an expression for $w_{\tau\rho_0}$ that differs from (27). Specifically

$$w_{\tau\rho_0} = \left\{\frac{-\sigma}{(1+\sigma)^2}, \frac{1}{(1+\sigma)^2}, 0\right\}.$$

The remaining coefficients: $\alpha_{\tau\rho_0}$, $\alpha_{\tau\tau_0}$, and $w_{\tau\tau_0}$ are unchanged. To second order, the center manifold is given by

$$x_{cm} = \left\{\frac{(1+\sigma)^2-(\rho-1)\sigma}{(1+\sigma)^2}\tilde{\tau}, \frac{(1+\sigma)^2+(\rho-1)}{(1+\sigma)^2}\tilde{\tau}, \tilde{\tau}^2/\beta\right\}.$$

This agrees with the expression obtained by the usual graph procedure [2, 3]. The parameterizations in terms of τ and $\tilde{\tau}$ are related by the transformation

$$\tau = \tilde{\tau}\left(\frac{2(1+\sigma)^2-(\sigma-1)(\rho-1)}{2(1+\sigma)^2} + \mathcal{O}((\rho-1)^2)\right)$$

as can be verified by substituting into (30) and retaining terms of order $\tilde{\tau}$, $\tilde{\tau}^2$, and $\tilde{\tau}(\rho-1)$.

5 Summary

We have given a technique to identify the center manifold, and the flow on the center manifold in a coordinate-independent manner. This is accomplished by writing the center manifold as an embedding, rather than as a graph over the center subspace. With this technique it is unnecessary to transform the original system to block-diagonal form. This is an advantage for computer algebra implementation.

References

[1] John Guckenheimer and Philip Holmes. *Non-linear Oscillations, Dynamical Systems and Bifurcations of Vector Fields.* Springer-Verlag, New York, 1983.

[2] Richard H. Rand and Dieter Armbruster. *Perturbation Methods, Bifurcation Theory and Computer Algebra.* Springer-Verlag, New York, 1987.

[3] S. Wiggins. *Introduction to Applied Nonlinear Dynamical Systems and Chaos.* Springer-Verlag, New York, 1990.