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A Coordinate-Independent Center Manifold Reduction

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Abstract

We give a method for performing the center manifold reduction that eliminates the need to transform the original equations of motion into eigencoordinates. To achieve this, we write the center manifold as an embedding, rather than as a graph over the center subspace.

1 Introduction

The center manifold reduction is a technique for eliminating non-essential degrees of freedom in bifurcation problems. The low-dimensional equations of motion on the center manifold, or their projection onto the center subspace, tell us about the flow in the vicinity of the bifurcation point.

This paper provides an alternative to the graph construction commonly used to identify the center manifold [1, 2, 3, for example]. Instead, we construct the center manifold as an embedding. The technique eliminates the need to transform the original equations of motion into eigencoordinates, thus emphasizing that the center manifold is a geometric object whose specification does not require particular coordinates.

2 The Graph Construction

Consider a vector field $f(X) : R^N \to R^N$ with an equilibrium at the origin. Let Df_0 be the linear part of the vector field at this equilibrium. The center subspace E^c is the space spanned by the (generalized) eigenvectors of Df_0 corresponding to eigenvalues with zero real part. The center manifold is tangent to E^c at $X = 0$ and is invariant under the flow of f.

In the usual center manifold reduction, one transforms the equations of motion into eigen-coordinates. One then writes the center manifold as a graph over the center subspace, the latter having been linearly decoupled from the other degrees of freedom by the coordinate transformation. [1, 2].

In the eigen-coordinates, the system of differential equations has the form [1, p. 130]),

$$\begin{aligned} \dot{x} &= Bx + F(x, y) \\ \dot{y} &= Cy + G(x, y) \end{aligned} \tag{1}$$

where B has eigenvalues with real part equal to zero and C has eigenvalues with negative real part[1]. The functions F and G, and their first derivatives, vanish at the origin.

The center manifold is written as a graph

$$W^c = \{ \ (x, y) \mid y = h(x) \ \}, \quad h(0) = Dh(0) = 0 \tag{2}$$

in the neighborhood of the origin. The boundary conditions $h(0) = 0$ and $Dh(0) = 0$ insure that the center manifold passes through the equilibrium and is tangent to E^c at the equilibrium. Since the center manifold is invariant under the flow, one can substitute $y = h(x)$ into the second equation of (1) and obtain

$$Dh(x) \, [\, Bx + F(x, h(x)) \,] \ - \ Ch(x) \ - \ G(x, h(x)) \ = \ 0 \tag{3}$$

This is solved for $h(x)$ by expanding in a power series about the origin. With the center manifold identified, the vector field on $y = h(x)$ is projected onto the center subspace

$$\dot{x} \ = \ Bx + F(x, h(x)). \tag{4}$$

The stability of the equilibrium for the full system (1) is given by the stability for the reduced system (4).

3 The Center Manifold as an Embedding

The above procedure requires an initial transformation to eigencoordinates. For high-dimensional systems this (block) diagonalization is prohibitive for hand calculation and one turns to a machine implementation. Unfortunately, diagonalizing large algebraic matrices can be difficult for symbolic computation systems.

We give an alternative center manifold reduction that dispenses with the need to perform the initial coordinate transformation. The procedure requires only knowledge of the vectors spanning the center subspace. For systems where Df_0 has zero eigenvalues, the center subspace is just the kernel of Df_0. In our experience, symbolic computation packages are able to find kernels of matrices with little difficulty.

[1] We assume that the unstable subspace is empty, though this is not necessary for the construction.

We assume that the original system of equations

$$\dot{X} = f(X) \tag{5}$$

has an equilibrium at the origin; $f(0) = 0$. For simplicity we assume that Df_0 has a one-dimensional kernel spanned by v_r. It is straightforward to extend the procedure to multi-dimensional kernels.

Before proceeding, we need to establish some notation. We denote by $Df[a]$ the action of Df on the vector a. The result is a vector with components

$$(Df[a])^i \equiv \sum_{j=1}^{N} \frac{\partial f^i}{\partial X^j} a^j.$$

where a^i denotes the i^{th} component of the vector a. Similarly $D^2 f[a, b]$ is the action of the second derivative $D^2 f$ on the pair of vectors a, b. The result is a vector with components

$$(D^2 f[a, b])^i \equiv \sum_{j,k} \frac{\partial^2 f^i}{\partial X^j \partial X^k} a^j b^k.$$

Contractions of higher order derivatives are similarly defined.

We write the center manifold as an embedding from $R^1 \to R^N$,

$$X_{cm}(\tau) = v_r \tau + w(\tau) = v_r \tau + w_\tau(0) + \frac{1}{2} w_{\tau\tau}(0) \tau^2 + \dots \tag{6}$$

having expanded the embedding function $w(\tau)$ in a Taylor series about $\tau = 0$, and denoted derivatives of w with respect to τ by subscripts. The embedding function satisfies the boundary conditions

$$w(0) = 0, \quad w_\tau(0) \equiv w_{\tau\,0} = 0 \quad \text{with} \quad w \perp v_r \ . \tag{7}$$

Here we choose w to lie in the orthogonal complement to v_r. We could use another splitting, e.g. choose w in the range of Df_0. This leads to a different parameterization of the center manifold, that is $w(\tau)$ will have a different functional form. This is illustrated in the example in §4.

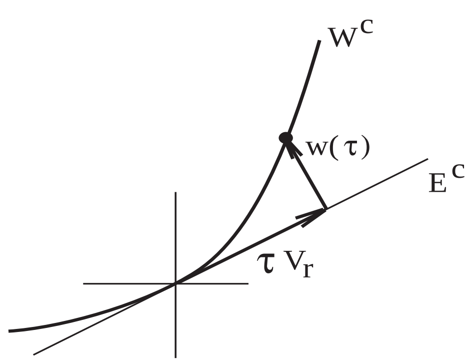

Figure 1: Center manifold W^c and center subspace E^c.

The first condition in (7), along with the form of the embedding (6), insures that the center manifold passes through the equilibrium at the origin. The second condition in (7) insures that the center manifold is tangent to the center subspace E^c at the equilibrium. The geometry of the construction is shown in Figure 1.

Since the center manifold is invariant under the flow, the vector field at any point on X_{cm} *must be tangent* to X_{cm}. Hence we can write

$$\begin{aligned} f(X_{cm}(\tau)) &= \alpha(\tau)\,\frac{dX_{cm}}{d\tau} \\ &= \alpha(\tau)\,(\,v_r + w_\tau(\tau)\,) \end{aligned} \tag{8}$$

where $\alpha(\tau)$ is a real-valued function. Note that since $f(0) = 0$,

$$\alpha(0) = 0 \ . \tag{9}$$

3.1 Flow on the Center Manifold

Points on the center manifold evolve under the flow according to

$$\dot{X}_{cm} \;=\; \frac{dX_{cm}}{d\tau}\,\dot{\tau} \;=\; f(\,X_{cm}(\tau)\,) \;=\; \frac{dX_{cm}}{d\tau}\,\alpha(\tau), \tag{10}$$

having used (8) in the last equality. Thus

$$\dot{\tau} \;=\; \alpha(\tau) \ ; \tag{11}$$

the scaling function $\alpha(\tau)$ gives the time rate of change of the embedding parameter τ induced by the motion on the center manifold.

3.2 Solving the Embedding Equation

The embedding function $w(\tau)$ and the scaling function $\alpha(\tau)$ are found by solving the tangency condition (8) in a Taylor series expansion about $\tau = 0$. For the first order term, take the derivative of (8) with respect to τ

$$Df\,[\,\frac{dX_{cm}}{d\tau}\,] \;=\; Df\,[\,v_r + w_\tau\,] \;=\; \alpha_\tau\,(v_r + w_\tau) + \alpha\,w_{\tau\tau}. \tag{12}$$

Evaluating (12) at $\tau = 0$ using (7) and (9) and the fact that $Df_0\,[\,v_r\,] = 0$ gives

$$\alpha_{\tau_0} \equiv \alpha_\tau(0) \;=\; 0 \ . \tag{13}$$

This expresses the fact that the linear part of the motion along the center manifold vanishes at the origin. For the second order term, we take the derivative of (12) with respect to τ leaving

$$\begin{aligned} Df\,[\,w_{\tau\tau}\,] \;&+\; D^2 f\,[v_r + w_\tau,\; v_r + w_\tau\,] \\ &= \alpha_{\tau\tau}\,(v_r + w_\tau) \;+\; 2\,\alpha_\tau\,w_{\tau\tau} \;+\; \alpha\,w_{\tau\tau\tau}. \end{aligned} \tag{14}$$

Evaluating this at $\tau = 0$ leaves

$$Df_0\,[\,w_{\tau\tau_0}\,] \;+\; D^2 f_0\,[v_r, v_r] \;=\; \alpha_{\tau\tau_0}\, v_r \tag{15}$$

Now let v_l be the span of the left kernel of Df_0. Left multiply (15) by v_l and solve for $\alpha_{\tau\tau_0}$ to obtain

$$\alpha_{\tau\tau_0} \;=\; v_l \cdot D^2 f_0[v_r, v_r] \;/\; (v_l \cdot v_r). \tag{16}$$

With this expression for $\alpha_{\tau\tau_0}$, (15) can be solved for $w_{\tau\tau_0}$

$$w_{\tau\tau_0} \;=\; L^{-1}\,(\,\alpha_{\tau\tau_0}\, v_r \;-\; D^2 f_0\,[\,v_r, v_r\,]\,) \tag{17}$$

where L is the restriction of Df_0 to $v_r^{\perp}$, and L^{-1} is its inverse.[2]

The procedure can be extended to arbitrary order. For reference the third order terms are

$$\alpha_{\tau\tau\tau_0} \;=\; \frac{v_l \cdot (\,3\,D^2 f_0[v_r, w_{\tau\tau_0}] + D^3 f_0[v_r, v_r, v_r] - 3\alpha_{\tau\tau_0}(v_l \cdot w_{\tau\tau_0})\,)}{v_l \cdot v_r} \tag{18}$$

$$w_{\tau\tau\tau_0} \;=\; L^{-1}(\alpha_{\tau\tau\tau_0}, v_r + 3\alpha_{\tau\tau_0} w_{\tau\tau_0} - 3D^2 f_0[w_{\tau\tau_0}, v_r] - D^3 f_0[v_r, v_r, v_r]). \tag{19}$$

We note that to obtain α to third order (and hence the motion on the center manifold to third order), we need only compute w to second order. This holds similarly for higher order calculations of α.

The reader familiar with the Liapunov-Schmidt (LS) reduction [4] will note a similarity between the computations involved in that technique, and those in equations (12) – (19). Both the center manifold reduction and the LS reduction are used to analyze bifurcations of equilibria. The LS reduction provides a low-dimensional, or reduced, function whose zeroes are in one-one *correspondence* with the zeroes of the vector field f. However unlike the algorithm given here, the LS reduction does not locate the center manifold.

4 An Example: The Lorenz Equations

The Lorenz equations

$$\begin{aligned} \dot{x} &= \sigma\,(y - x) \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= xy - \beta z \end{aligned}$$

have an equilibrium at $(x, y, z) = (0, 0, 0)$ with linearization

$$Df_0 = \begin{pmatrix} -\sigma & \sigma & 0 \\ \rho & -1 & 0 \\ 0 & 0 & -\beta \end{pmatrix}.$$

[2] The map $L : v_r^{\perp} \to \text{Range}\,(\,Df_0\,)$ is invertible. From (15), $\alpha_{\tau\tau}\, v_r - D^2 f_0\,[\,v_r, v_r\,]$ is in Range $(\,Df_0\,)$. Consequently $w_{\tau\tau_0}$ is uniquely defined in $v_r^{\perp}$.

At $\rho = 1$, Df_0 has right and left kernels $v_r = \{1, 1, 0\}$ and $v_l = \{1/\sigma, 1, 0\}$ respectively.

We construct the center manifold as an embedding from (τ, ρ) into (x, y, z, ρ)

$$X_{cm} \;=\; v_r\,\tau \;+\; w(\tau,\,\rho) \quad \text{with} \quad w \perp v_r \tag{20}$$

subject to the tangency condition

$$f(\,X_{cm}(\tau,\rho), \rho\,) \;=\; \alpha\,(\tau,\rho)\;\;(\,v_r + w_\tau(\tau,\rho)\,)\;\;. \tag{21}$$

Differentiating this with respect to ρ leaves

$$Df\,[\,w_\rho\,] + f_\rho \;=\; \alpha_\rho\,(\,v_r + w_\tau\,) \;+\; \alpha\,w_{\tau\rho}\;\;, \tag{22}$$

where $f_\rho \equiv \partial f/\partial\rho$. Evaluating the derivatives at the bifurcation point $(\tau, \rho) = (0, 1)$ and dotting with v_l gives

$$\alpha_{\rho\,0} = 0.$$

Substituting this result into (22) and evaluating the result at the bifurcation point leaves

$$Df_0\,[\,w_{\rho\,0}\,] = 0.$$

Since $w \perp v_r$, this requires that

$$w_{\rho\,0} = 0.$$

To obtain the $\tau\rho$ terms, we differentiate (22) with respect to τ :

$$D^2 f\,[\,w_\rho, v_r + w_\tau\,] \;+\; Df\,[\,w_{\rho\tau}\,] \;+\; Df_\rho\,[\,v_r + w_\tau\,] \;=$$

$$\alpha_{\tau\rho}\,(\,v_r + w_\tau\,) \;+\; \alpha_\tau\,w_{\tau\rho} \;+\; \alpha_\rho\,w_{\tau\tau} \;+\; \alpha\,w_{\tau\tau\rho}. \tag{23}$$

Evaluating this at the bifurcation point and solving for $\alpha_{\tau\rho\,0}$ and $w_{\tau\rho\,0}$, we find

$$\alpha_{\tau\rho\,0} \;=\; \frac{v_l \cdot (\,Df_{\rho\,0}\,[\,v_r\,]\,)}{v_l \cdot v_r} \;=\; \frac{\sigma}{1+\sigma} \tag{24}$$

$$w_{\tau\rho\,0} \;=\; L^{-1}\,(\,\alpha_{\tau\rho\,0}\,v_r - Df_{\rho\,0}\,[\,v_r\,]\,) \;=\; \left\{ \frac{-1}{2(1+\sigma)}, \frac{1}{2(1+\sigma)}, 0 \right\}\;. \tag{25}$$

The terms second order in τ are given by (16) and (17),

$$\alpha_{\tau\tau\,0} \;=\; 0 \tag{26}$$
$$w_{\tau\tau\,0} \;=\; \{0, 0, 2/\beta\}. \tag{27}$$

Thus to second order the center manifold is given by

$$\begin{aligned} X_{cm}(\tau,\rho) \;&=\; v_r\,\tau \;+\; w_{\tau\rho\,0}\,\tau(\rho-1) \;+\; \frac{1}{2} w_{\tau\tau\,0}\,\tau^2 \\ &=\; \left\{ \left(1 - \frac{\rho-1}{2(1+\sigma)}\right)\tau,\;\; \left(1 + \frac{\rho-1}{2(1+\sigma)}\right)\tau,\;\; \frac{\tau^2}{\beta} \right\}. \end{aligned} \tag{28}$$

To calculate the motion on the center manifold, we need the coefficient $\alpha_{\tau\tau\tau\,0}$ from (18). Since $D^3 f$ and $\alpha_{\tau\tau\,0}$ are both zero, we have

$$\alpha_{\tau\tau\tau\,0} \;=\; \frac{3v_l \cdot D^2 f_{\,0}[v_r, w_{\tau\tau\,0}]}{v_l \cdot v_r} \;=\; \frac{-6\sigma}{\beta(1+\sigma)}.$$

Finally, the term $\alpha_{\tau\tau\rho\,0}$ is shown to be zero by differentiating (23) with respect to τ, evaluating the resulting expression at the bifurcation point, and dotting with v_l.

Including terms of order $\tau(\rho-1)$, τ^2, $\tau^2(\rho-1)$, and τ^3, the motion on the center manifold is given by

$$\begin{aligned} \dot{\tau} \;=\; \alpha(\tau) \;&=\; \frac{1}{3!}\alpha_{\tau\tau\tau\,0}\,\tau^3 \;+\; \alpha_{\tau\rho\,0}\,\tau\,(\rho-1) \;+\; \ldots \\ &=\; -\frac{\sigma}{\beta(1+\sigma)}\,\tau^3 \;+\; \frac{\sigma}{1+\sigma}\,\tau\,(\rho-1)\ldots \end{aligned} \tag{29}$$

which is the normal form for a super-critical pitchfork bifurcation.

4.1 Alternative Parameterization

In the construction above, we wrote the center manifold in terms of components in E^c and $E^{c\perp}$ as depicted in Figure 1. The constraint $w(\tau) \perp v_r$ uniquely defines the preimage of $Df_{\,0}$ used to solve for the coefficients in the series expansion of $w(\tau)$ (e.g. (17) and (19)). One can, however, use any other convenient decomposition. For example we can require $w(\tau) \in \mathrm{Range}(Df_{\,0})$ as in Figure 2. This defines a different preimage of $Df_{\,0}$ and leads to different functional forms for $w(\tau)$ and $\alpha(\tau)$. These differences amount to a reparameterization of the center manifold.

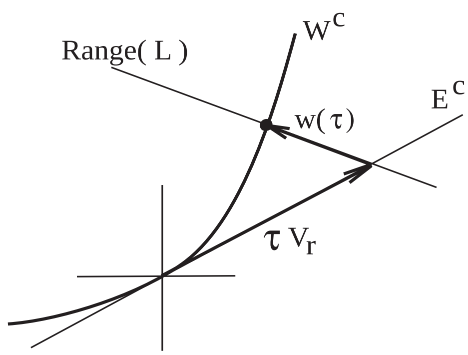

Figure 2: Alternative parameterization of the center manifold.

For the example of the Lorenz system, writing the center manifold as

$$\tilde{X}_{cm}(\tilde{\tau},\ \rho) \;=\; v_r\,\tilde{\tau} \;+\; \tilde{w}(\tilde{\tau},\rho), \qquad \tilde{w} \in \mathrm{Range}\,(Df_{\,0}) \tag{30}$$

results in a $\tilde{w}_{\tilde{\tau}\rho\,0}$ that differs from (25). Specifically, we find

$$\tilde{w}_{\tilde{\tau}\rho\,0} \;=\; \left\{ \frac{-\sigma}{(1+\sigma)^2},\ \frac{1}{(1+\sigma)^2},\ 0 \right\}.$$

The remaining coefficients, $\alpha_{\tilde{\tau}\tilde{\tau}\,0}$, $w_{\tilde{\tau}\tilde{\tau}\,0}$, and $\alpha_{\tilde{\tau}\rho\,0}$ are the same as (26), (27), and (24) respectively. To second order, the center manifold is given by

$$X_{cm} \;=\; \left\{ \left(1 - \frac{(\rho-1)\,\sigma}{(1+\sigma)^2}\right)\,\tilde{\tau},\;\; \left(1 + \frac{(\rho-1)\,\sigma}{(1+\sigma)^2}\right)\,\tilde{\tau},\;\; \frac{\tilde{\tau}^2}{\beta} \right\}. \tag{31}$$

This agrees with the expression obtained by the usual graph procedure [2, 3]. The parameterizations in terms of τ and $\tilde{\tau}$ are related by the transformation

$$\tau \;=\; \tilde{\tau}\,\left(\frac{2\,(1+\sigma)^2 - (\sigma-1)\,(\rho-1)}{2\,(1+\sigma)^2} \;+\; \mathcal{O}(\,(\rho-1)^2\,) \right) \tag{32}$$

as can be verified by substituting into (28) and retaining terms of order $\tilde{\tau}$, $\tilde{\tau}^2$, and $\tilde{\tau}(\rho-1)$.

5 Summary

We have presented a technique to identify the center manifold, and the flow on the center manifold, in a coordinate-independent manner. This is accomplished by writing the center manifold as an embedding, rather than as a graph over the center subspace. With this technique it is not necessary to transform the original system to coordinates for which the linear part of the vector field is block-diagonal. This provides a practical advantage for computation.

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