Improving Programs which Recurse over Multiple Inductive Structures*

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Abstract

This paper considers generic recursion schemes for programs which recurse over multiple inductive structures simultaneously, such as equality, zip and the nth element of a list function. Such schemes have been notably absent from previous work. This paper defines a uniform mechanism for defining such programs and shows that these programs satisfy generic theorems. These theorems are the basis for an automatic improvement algorithm. This algorithm is an improvement over the algorithm presented earlier [15] because, in addition to inducting over multiple structures, it can be incorporated into any algebraic language and is no longer restricted to a "safe" subset.

1 Introduction

In previous work [15, 16] we have shown how programming algebraically with *generic recursion schemes* provides a theory amenable to program calculation [14]. This theory provides a basis for automatic optimization techniques which capture many well-known transformations. Unfortunately, these recursion schemes may induct over only one structure at a time, and thus cannot capture such common functions as *structural equality*, *zip*, or the function that computes the *nth* element of a list.

In this paper we generalize the generic reduction scheme to induct over any number of structures (not necessarily of the same type) simultaneously. We show that this induction scheme has a generic promotion theorem, and that the theorem supports a normalization algorithm that automatically calculates a number of previously unrelated improvements to programs, such as deforestation, loop fusion, and partial evaluation. This is an important step towards an automatic optimization phase in compilers of algebraic programs.

Recursion is the *Goto* of functional programming. Languages which allow arbitrary recursive programs lack the structure necessary for automatic optimization. Much recent research has focused on the design of programming systems whose control structures are exclusively some generic recursion schemes [3, 10, 11]. We call this programming style *algebraic programming* because of its reliance on algebras and combinators for encoding control. Notable absent from these works are generic recursion schemes for inducting over multiple structures.

The goal of this paper is to describe internal program representations for algebraic programs which are amenable to completely automatic optimization and transformation methods, yet are expressive enough to encode algorithms which induct over multiple structures simultaneously. These program representations are suitable for use in the back-end of a compiler for an algebraic programming language. In this work we do not consider user interfaces to algebraic programming languages, but rather demonstrate that algebraic

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recursion schemes are a practical internal form for representing programs and describe a single automatic mechanism for performing a wide range of optimizations over such programs.

1.1 Motivation and Background

Consider for example the following inductive type equation that captures natural numbers:

$$\text{nat} \quad = \quad \text{Zero} \mid \text{Succ } \textbf{of } \text{nat}$$

The *reduction* over a *nat* k can be computed by $\text{red}^{nat}(f_z, f_s)\, k$, where f_z and f_s are functions associated with the value constructors Zero and Succ. The combinator $\text{red}^{nat}(f_z, f_s)$ is defined as follows:

$$\begin{array}{lcl} \text{red}^{nat}(f_z, f_s)\, \text{Zero} & = & f_z() \\ \text{red}^{nat}(f_z, f_s)\, (\text{Succ}(n)) & = & f_s(\text{red}^{nat}(f_z, f_s)\, n) \end{array}$$

Functions f_z and f_s are called *accumulating functions.*

For example:

$$x + y \quad = \quad \text{red}^{nat}(\lambda().y, \lambda(r).\text{Succ}(r))\, x$$

The variable r in $\lambda(r)$ is the partial result of the reduction, since it represents the value of the recursive call $\text{red}^{nat}(f_z, f_s)\, n$. It is called an *accumulative result variable.*

The reduction operator can be generalized for most inductively defined data type. One such type is *list*:

$$\text{list}(\alpha) \quad = \quad \text{Nil} \mid \text{Cons } \textbf{of } \alpha \times \text{list}(\alpha)$$

where α is a type variable. The reduction for lists is very similar to the reduction for natural numbers:

$$\begin{array}{lcl} \text{red}^{list}(f_n, f_c)\, \text{Nil} & = & f_n() \\ \text{red}^{list}(f_n, f_c)\, (\text{Cons}(a, l)) & = & f_c(a, \text{red}^{list}(f_n, f_c)\, l) \end{array}$$

For example,

$$\text{length}(x) \quad = \quad \text{red}^{list}(\lambda().\text{Zero}, \lambda(a, r).\text{Succ}(r))\, x$$

For each such inductive data type there is a generic theorem called the *promotion theorem* [13, 14]. For lists this theorem takes the following form:

$$\frac{\begin{array}{lcl} \phi_n() & = & g(f_n()) \\ \phi_c(a, g(r)) & = & g(f_c(a, r)) \end{array}}{g(\text{red}^{list}(f_n, f_c)\, x) \;=\; \text{red}^{list}(\phi_n, \phi_c)\, x}$$

This states that the application of any unary function g to a reduction over a list x is another reduction over the same list x whose accumulating functions ϕ_n and ϕ_c are related to the original accumulating functions f_n and f_c by the two equations in the premise of the theorem. The first equation calculates directly a solution for ϕ_n in terms of g and f_n. The second equation, though, cannot be solved directly. This equation must be rearranged so that the term $g(r)$ can be generalized to a variable in both sides of the equation. That is, the term $g(f_c(a, r))$ must be transformed into a form $\mathcal{T}(g(r))$, for some term $\mathcal{T}$ that depends on $g(r)$ but not on r. Such work was previously reported in [15, 7, 6, 5] where this transformation is achieved by the generalization phase of the *normalization algorithm.*

As an example, we will improve length(append(x, y)), where

$$\begin{array}{lcll} \text{length}(x) & = & \text{red}^{list}(\lambda().\text{Zero}, \lambda(a, r).\text{Succ}(r))\, x & \\ \text{append}(x, y) & = & \text{red}^{list}(f_n, f_c)\, x & \text{where } \left\{ \begin{array}{l} f_n \;=\; \lambda().y \\ f_c \;=\; \lambda(a, r).\text{Cons}(a, r) \end{array} \right. \end{array}$$

We need to find some $\text{red}^{list}(\phi_n, \phi_c)\, x = \text{length}(\text{append}(x, y))$. We apply the *list* promotion theorem with $g = \text{length}$ and $\text{red}^{list}(f_n, f_c)\, x = \text{append}(x, y)$:

1) $\phi_n() \quad = \quad g(f_n()) \;=\; \text{length}(f_n()) \;=\; \text{length}(y)$

2) $$\begin{array}{lcll} \phi_c(a, \text{length}(r)) & = & g(f_c(a, r)) \;=\; \text{length}(f_c(a, r)) & \\ & = & \text{length}(\text{Cons}(a, r)) & \\ & = & \text{Succ}(\text{length}(r)) & \textit{by the length definition} \\ \Rightarrow\; \phi_c(a, u) \;=\; \text{Succ}(u) & & & \textit{where } \text{length}(r) \textit{ was generalized to } u \end{array}$$

Therefore, length(append(x, y)) is transformed into:

$$\text{red}^{list}(\lambda().\text{length}(y), \lambda(a, u).\text{Cons}(a, u))\, x$$

Note that length(append(x, y)) generates an intermediate data structure (a list), since append(x, y) constructs a list which is consumed by length. This data structure is not produced when this composition is normalized into the reduction above.

Even though the reduction scheme that inducts over one value (henceforth called a unary reduction) is very powerful, there are still some important functions that cannot be captured by this mechanism. One class are binary functions such as structural equality. For example, the structural equality over lists is defined as follows:

$$\begin{array}{lcl} \text{listeq}(\text{Nil}, \text{Nil}) & = & \text{True} \\ \text{listeq}(\text{Nil}, \text{Cons}(b, s)) & = & \text{False} \\ \text{listeq}(\text{Cons}(a, l), \text{Nil}) & = & \text{False} \\ \text{listeq}(\text{Cons}(a, l), \text{Cons}(b, s)) & = & (a = b) \wedge \text{listeq}(l, s) \end{array}$$

Function listeq cannot be expressed as a unary reduction since it needs to walk through the two input lists simultaneously. Instead we need a new recursion scheme $F = \text{red}^{list \times list}(f_{nn}, f_{nc}, f_{cn}, f_{cc})$ defined as follows*:

$$\begin{array}{lcl} F(\text{Nil}, \text{Nil}) & = & f_{nn}((), ()) \\ F(\text{Nil}, \text{Cons}(b, s)) & = & f_{nc}((), (b, s)) \\ F(\text{Cons}(a, l), \text{Nil}) & = & f_{cn}((a, l), ()) \\ F(\text{Cons}(a, l), \text{Cons}(b, s)) & = & f_{cc}(a, (b, F(l, s))) \end{array}$$

In that case we have:

$$\begin{array}{lcl} \text{listeq}(x, y) & = & \text{red}^{list \times list}(\lambda((), ()).\text{True}, \lambda((), (b, s)).\text{False}, \lambda((a, l), ()).\text{False}, \\ & & \qquad \lambda(a, (b, r)).(a = b) \wedge r)\, (x, y) \end{array}$$

We call $\text{red}^{list \times list}$ a binary reduction because it inducts over two data structures simultaneously.

This paper extends the theory of unary reductions to capture all reduction schemes that induct over any number of data structures, which need not necessarily be of the same type. We will present and prove a very general promotion theorem that captures all types of compositions between these general reduction schemes. We will use this theorem as the basis of a very effective and efficient normalization algorithm that improves any safe program in our term language. This algorithm eliminates intermediate data structures, which might be generated when reductions are nested, passing intermediate results from one to another.

This paper is organized as follows. Section 2 reviews the definition of the unary reduction operator for any inductively defined data type. These combinators are then generalized in such a way that they can be used for defining reductions that induct over multiple data types. Section 3 presents two instances of the promotion theorem, one is for promoting a unary function and the other for promoting a binary function. The promotion theorem in its most general form is presented in the Appendix. Section 4 presents the normalization algorithm.

2 Reductions

The type definitions considered in this paper are the inductive types defined by using recursive equations of the form:

$$T(\alpha_1, \ldots, \alpha_p) \quad = \quad C_1 \textbf{ of } t_1 \;\mid\; \cdots \;\mid\; C_n \textbf{ of } t_n$$

where $\alpha_1, \ldots, \alpha_p$ denote type variables (abbreviated by the vector $\overline{\alpha}$), the C_i are names of value constructor functions, and each type t_i is an inductive subcomponent of type $T(\overline{\alpha})$. An *inductive subcomponent* of a type $T(\overline{\alpha})$ has one of the following forms:

α_i	a type variable in the set $\alpha_1, \ldots, \alpha_p$
$T(\overline{\alpha})$	the recursive reference to $T(\overline{\alpha})$
$t_1 \times t_2$	a pair of two inductive subcomponents of $T(\overline{\alpha})$
$S(t_1, \ldots, t_q)$	where S is a previously defined inductive type constructor and each t_i is an inductive subcomponent of $T(\overline{\alpha})$

*The unintuitive "shape" of the domain of the accumulating functions will be explain later.

For example, the following are inductive type definitions:

$$\begin{array}{rcl} \text{boolean} & = & \text{False} \mid \text{True} \\ \text{list}(\alpha) & = & \text{Nil} \mid \text{Cons } \mathbf{of}\ \alpha \times \text{list}(\alpha) \\ \text{tree}(\alpha,\beta) & = & \text{Tip } \mathbf{of}\ \alpha \mid \text{Node } \mathbf{of}\ \beta \times \text{tree}(\alpha,\beta) \times \text{tree}(\alpha,\beta) \\ \text{bush}(\alpha) & = & \text{Leaf } \mathbf{of}\ \alpha \mid \text{Branch } \mathbf{of}\ \text{list}(\text{bush}(\alpha)) \end{array}$$

Definition 1 (The Functor E) *For each value constructor C of type $t \rightarrow T(\overline{\alpha})$ we associate a functor E_c^T, such that $E_c^T(f) = \mathcal{K}[T(\overline{\alpha}), t](f)$. The combinator $\mathcal{K}$ is defined by the following inductive equations:*

$$\begin{array}{lcl} \mathcal{K}[T(\overline{\alpha}), \alpha_i](f) & = & \text{id} \\ \mathcal{K}[T(\overline{\alpha}), T(\overline{\alpha})](f) & = & f \\ \mathcal{K}[T(\overline{\alpha}), t_1 \times t_2](f) & = & \mathcal{K}[T(\overline{\alpha}), t_1](f) \times \mathcal{K}[T(\overline{\alpha}), t_2](f) \\ \mathcal{K}[T(\overline{\alpha}), S(t_1, \ldots, t_q)](f) & = & \text{map}^S(\mathcal{K}[T(\overline{\alpha}), t_1](f), \ldots, \mathcal{K}[T(\overline{\alpha}), t_q](f)) \end{array}$$

where id *is the identity function, the product of two functions g and h is defined by $(g \times h)(x, y) = (g\,x, h\,y)$, and* map^S *is a map over the inductive type $S(\alpha_1, \ldots, \alpha_q)$, that is, it maps the parametric type $S(\alpha_1, \ldots, \alpha_q)$ into the type $S(\beta_1, \ldots, \beta_q)$.*

It is easy to prove that $E_c^T(\text{id}) = \text{id}$ and $E_c^T(f \circ g) = E_c^T(f) \circ E_c^T(g)$, where $\circ$ is function composition (i.e. $(f \circ g)\,x = f(g(x))$). That is, E_c^T is a functor.

For example:

$$\begin{array}{llll} E_{Nil}(f) & = \text{id} & = & \lambda().() \\ E_{Cons}(f) & = \text{id} \times f & = & \lambda(a, s).(a, f(s)) \\ E_{Node}(f) & = \text{id} \times f \times f & = & \lambda(i, l, r).(i, f(l), f(r)) \\ E_{Branch}(f) & = \text{map}^{list}(f) & = & \lambda(l).\text{map}^{list}(f)\,l \end{array}$$

Definition 2 (Unary Reduction) *The unary reduction operator over the type T is $\text{red}^T(\overline{f})$ and it is defined by the following set of recursive equations, one for each value constructor C of T:*

$$\text{red}^T(\overline{f}) \circ C = f_c \circ E_c^T(\text{red}^T(\overline{f}))$$

where variable $\overline{f}$ is the vector of all accumulating functions f_c for each value constructor C of T.

For example, the *tree* reduction operator is defined as:

$$\begin{array}{lcl} \text{red}^{tree}(f_t, f_n)\,(\text{Tip}(a)) & = & f_t(a) \\ \text{red}^{tree}(f_t, f_n)\,(\text{Node}(b, l, r)) & = & f_n(b, \text{red}^{tree}(f_t, f_n)\,l, \text{red}^{tree}(f_t, f_n)\,r) \end{array}$$

The *bush* reduction operator is defined as:

$$\begin{array}{lcl} \text{red}^{bush}(f_l, f_b)\,(\text{Leaf}(a)) & = & f_l(a) \\ \text{red}^{bush}(f_l, f_b)\,(\text{Branch}(l)) & = & f_b(\text{map}^{list}(\text{red}^{bush}(f_l, f_b))\,l) \end{array}$$

The following are some examples of computations that use unary reduction operators:

$$\begin{array}{lcl} \text{append}(x, y) & = & \text{red}^{list}(\lambda().y, \lambda(a, r).\text{Cons}(a, r))\,x \\ \text{length}(x) & = & \text{red}^{list}(\lambda().\text{Zero}, \lambda(a, r).\text{Succ}(r))\,x \\ \text{reverse}(x) & = & \text{red}^{list}(\lambda().\text{Nil}, \lambda(a, r).\text{append}(r, \text{Cons}(a, \text{Nil})))\,x \\ x + y & = & \text{red}^{nat}(\lambda().y, \lambda(r).\text{Succ}(r))\,x \\ \text{if } x \text{ then } y \text{ else } z & = & \text{red}^{boolean}(\lambda().z, \lambda().y)\,x \\ \text{sum\_tree}(x) & = & \text{red}^{tree}(\lambda(a).a, \lambda(b, m, n).b + m + n)\,x \\ \text{reflect\_bush}(x) & = & \text{red}^{bush}(\lambda(a).\text{Leaf}(a), \lambda(r).\text{Branch}(\text{reverse}(r)))\,x \end{array}$$

In order to generalize reduction to capture recursion schemas that traverse more than one data structure simultaneously, we need to generalize the functor E.

Definition 3 (Generalized Product Type) *A generalized product type τ is either an inductive type T or a pair $\tau_1 \times \tau_2$, where τ_1 and τ_2 are generalized product types.*

We use the symbol T for inductive types and τ for generalized product types.

Definition 4 (Generalized Constructor) *The set of generalized constructors $\mathcal{GC}(\tau)$ of a generalized product type τ is defined inductively using list comprehensions:*

$$\begin{array}{rcl} \mathcal{GC}(T) & = & [C \mid C \text{ is a value constructor of the inductive type } T] \\ \mathcal{GC}(\tau_1 \times \tau_2) & = & [c_1 \times c_2 \mid c_1 \leftarrow \mathcal{GC}(\tau_1),\, c_2 \leftarrow \mathcal{GC}(\tau_2)] \end{array}$$

We will use the symbols C, C_1, or C_2 for value constructors and the symbols c, c_1, or c_2 for generalized constructors.

For example, the generalized constructors for $\tau(\alpha) = \text{list}(\alpha) \times \text{nat}$ are: Nil$\times$Zero, Nil$\times$Succ, Cons$\times$Zero, and Cons$\times$Succ. The generalized constructors for $\tau(\alpha) = \text{boolean} \times (\text{list}(\alpha) \times \text{nat})$ are False$\times$(Nil$\times$Zero), False$\times$(Nil$\times$Succ), False$\times$(Cons$\times$Zero), False$\times$(Cons$\times$Succ), True$\times$(Nil$\times$Zero), True$\times$(Nil$\times$Succ), True$\times$(Cons$\times$Zero), and True$\times$(Cons$\times$Succ).

Definition 5 (Inductive Constructor) *A generalized constructor $c \in \mathcal{GC}(\tau)$ is* inductive *(denoted as* inductive(c)*) if either c is the value constructor C of type $t \rightarrowtail T$ and type t (the domain of C) contains a reference to T, or $c = c_1 \times c_2$ and both c_1 and c_2 are inductive constructors.*

That is, a generalized constructor c of τ is not inductive if the domain of any of its constituent value constructors $C : t \rightarrowtail T$ has no recursive reference to type T. For example, Cons$\times$Succ is inductive while Cons$\times$Zero is not.

The following combinators are defined in terms of the E functor and they are, in a way, generalizations of E:

Definition 6 (The Functor $\mathcal{D}$)

$$\begin{array}{lcll} \mathcal{D}^{\tau}_{c}(f) & = & \text{id} & \text{if } \neg\text{inductive}(c) \\ \left.\begin{array}{lcl} \mathcal{D}^{T}_{c}(f) & = & E^{T}_{c}(f) \\ \mathcal{D}^{\tau_1 \times \tau_2}_{c_1 \times c_2}(f) & = & \mathcal{D}^{\tau_1}_{c_1}(\mathcal{D}^{\tau_2}_{c_2}(f)) \end{array}\right\} & & \text{if inductive}(c) \end{array}$$

Note that $\mathcal{D}$ is a functor since it is a composition of functors.

Definition 7 (The Combinator $\mathcal{E}$)

$$\begin{array}{lcll} \mathcal{E}^{\tau}_{c}(f) & = & \text{id} & \text{if } \neg\text{inductive}(c) \\ \left.\begin{array}{lcl} \mathcal{E}^{T}_{c}(f) & = & E^{T}_{c}(f) \\ \mathcal{E}^{\tau_1 \times \tau_2}_{c_1 \times c_2}(f) & = & \lambda(x,y).\mathcal{E}^{\tau_1}_{c_1}(\lambda z.\mathcal{E}^{\tau_2}_{c_2}(\lambda w.f(z,w))\,y)\,x \end{array}\right\} & & \text{if inductive}(c) \end{array}$$

For example:

$$\begin{array}{lcll} \mathcal{D}_{Nil\times Tip}(f) & = & \text{id} & = \lambda((),i).((),i) \\ \mathcal{E}_{Nil\times Tip}(f) & = & \text{id} & = \lambda((),i).((),i) \\ \mathcal{D}_{Cons\times Tip}(f) & = & \text{id} & = \lambda((a,s),i).((a,s),i) \\ \mathcal{E}_{Cons\times Tip}(f) & = & \text{id} & = \lambda((a,s),i).((a,s),i) \end{array}$$

$$\begin{array}{lcl} \mathcal{D}_{Cons\times Cons}(f) & = & \lambda(x,(y,r)).(x,(y,f(r))) \\ \mathcal{E}_{Cons\times Cons}(f) & = & \lambda((x,xs),(y,ys)).(x,(y,f(xs,ys))) \\ \mathcal{D}_{Cons\times Succ}(f) & = & \lambda(x,r).(x,f(r)) \\ \mathcal{E}_{Cons\times Succ}(f) & = & \lambda((x,xs),n).(x,f(xs,n)) \end{array}$$

$$\begin{array}{lcl} \mathcal{D}_{Cons\times Node}(f) & = & \lambda(a,(i,r_1,r_2)).(a,(i,f(r_1),f(r_2))) \\ \mathcal{E}_{Cons\times Node}(f) & = & \lambda((a,s),(i,l,r)).(a,(i,f(s,l),f(s,r))) \\ \mathcal{D}_{Node\times Node}(f) & = & \lambda(i,(j,r_1,r_2),(k,r_3,r_4)).(i,(j,f(r_1),f(r_2)),(k,f(r_3),f(r_4))) \\ \mathcal{E}_{Node\times Node}(f) & = & \lambda((i,l,r),(j,m,n)).(i,(j,f(l,m),f(l,n)),(j,f(r,m),f(r,n))) \end{array}$$

Definition 8 (The Combinator $\mathcal{M}$)

$$\begin{array}{lcll} \mathcal{M}^{\tau}_{c}(f) & = & \text{id} & \text{if } \neg\text{inductive}(c) \\ \mathcal{M}^{\tau_1 \times \tau_2}_{c_1 \times c_2}(f) & = & \lambda(x,y).\mathcal{D}^{\tau_1}_{c_1}(\lambda z.\mathcal{D}^{\tau_2}_{c_2}(\lambda w.f(z,w))\,y)\,x & \text{if inductive}(c) \end{array}$$

Note that if τ_1 and τ_2 are the simple inductive types T_1 and T_2 respectively, then $\mathcal{M}^{T_1\times T_2}_{c_1\times c_2}(f) = \mathcal{E}^{T_1\times T_2}_{c_1\times c_2}(f)$.

Properties:

$$\begin{aligned}
\mathcal{D}^\tau_c(\mathrm{id}) &= \mathrm{id} && (1)\\
\mathcal{D}^\tau_c(f\circ g) &= \mathcal{D}^\tau_c(f)\circ\mathcal{D}^\tau_c(g) && (2)\\
\mathcal{E}^\tau_c(f\circ g) &= \mathcal{D}^\tau_c(f)\circ\mathcal{E}^\tau_c(g) && (3)\\
\mathcal{E}^{\tau_1\times\tau_2}_{c_1\times c_2}(g\circ(f\times h)) &= \mathcal{M}^{\tau_1\times\tau_2}_{c_1\times c_2}(g)\circ(\mathcal{E}^{\tau_1}_{c_1}(f)\times\mathcal{E}^{\tau_2}_{c_2}(h)) && (4)
\end{aligned}$$

Proof: Properties 1 and 2 are true because $\mathcal{D}$ is a functor (since it is a composition of functors). Property 3 is true for a non-inductive c and for $\tau = T$. We assume that it is true for $\tau = \tau_1$ and $\tau = \tau_2$ (induction hypothesis). Then for $\tau = \tau_1\times\tau_2$ and $c = c_1\times c_2$ we have:

$$\begin{aligned}
\mathcal{D}^\tau_c(f)\circ\mathcal{E}^\tau_c(g) &= \mathcal{D}^{\tau_1\times\tau_2}_{c_1\times c_2}(f)\circ\mathcal{E}^{\tau_1\times\tau_2}_{c_1\times c_2}(g)\\
&= \mathcal{D}^{\tau_1}_{c_1}(\mathcal{D}^{\tau_2}_{c_2}(f))\circ(\lambda(x,y).\mathcal{E}^{\tau_1}_{c_1}(\lambda z.\mathcal{E}^{\tau_2}_{c_2}(\lambda w.g(z,w))\,y)\,x) && \text{by Definitions 6 and 7}\\
&= \lambda(x,y).\mathcal{D}^{\tau_1}_{c_1}(\mathcal{D}^{\tau_2}_{c_2}(f))(\mathcal{E}^{\tau_1}_{c_1}(\lambda z.\mathcal{E}^{\tau_2}_{c_2}(\lambda w.g(z,w))\,y)\,x)\\
&= \lambda(x,y).\mathcal{E}^{\tau_1}_{c_1}((\mathcal{D}^{\tau_2}_{c_2}(f)\circ(\lambda z.\mathcal{E}^{\tau_2}_{c_2}(\lambda w.g(z,w))\,y))\,x) && \text{induction hypothesis}\\
&= \lambda(x,y).\mathcal{E}^{\tau_1}_{c_1}(\lambda z.\mathcal{D}^{\tau_2}_{c_2}(f)(\mathcal{E}^{\tau_2}_{c_2}(\lambda w.g(z,w))\,y)\,x)\\
&= \lambda(x,y).\mathcal{E}^{\tau_1}_{c_1}(\lambda z.\mathcal{E}^{\tau_2}_{c_2}((f\circ(\lambda w.g(z,w)))\,y)\,x) && \text{induction hypothesis}\\
&= \lambda(x,y).\mathcal{E}^{\tau_1}_{c_1}(\lambda z.\mathcal{E}^{\tau_2}_{c_2}(\lambda w.f(g(z,w)))\,y)\,x\\
&= \mathcal{E}^{\tau_1\times\tau_2}_{c_1\times c_2}(f\circ g) && \text{by Definition 7}\\
&= \mathcal{E}^\tau_c(f\circ g)
\end{aligned}$$

Property 4 can be proved as follows:

$$\begin{aligned}
&\mathcal{M}^{\tau_1\times\tau_2}_{c_1\times c_2}(g)\circ(\mathcal{E}^{\tau_1}_{c_1}(f)\times\mathcal{E}^{\tau_2}_{c_2}(h))\\
&\quad= (\lambda(x,y).\mathcal{D}^{\tau_1}_{c_1}(\lambda z.\mathcal{D}^{\tau_2}_{c_2}(\lambda w.g(z,w))\,y)\,x)\circ(\mathcal{E}^{\tau_1}_{c_1}(f)\times\mathcal{E}^{\tau_2}_{c_2}(h)) && \text{by Definition 8}\\
&\quad= \lambda(x,y).\mathcal{D}^{\tau_1}_{c_1}(\lambda z.\mathcal{D}^{\tau_2}_{c_2}(\lambda w.g(z,w))\,(\mathcal{E}^{\tau_2}_{c_2}(h)\,y))\,(\mathcal{E}^{\tau_1}_{c_1}(f)\,x) && \text{by beta reduction}\\
&\quad= \lambda(x,y).\mathcal{D}^{\tau_1}_{c_1}(\lambda z.\mathcal{E}^{\tau_2}_{c_2}(\lambda w.g(z,h\,w))\,y)\,(\mathcal{E}^{\tau_1}_{c_1}(f)\,x) && \text{by Property 3}\\
&\quad= \lambda(x,y).\mathcal{E}^{\tau_1}_{c_1}(\lambda z.\mathcal{E}^{\tau_2}_{c_2}(\lambda w.g(f\,z,h\,w))\,y)\,x && \text{by Property 3}\\
&\quad= \mathcal{E}^{\tau_1\times\tau_2}_{c_1\times c_2}(g\circ(f\times h)) && \text{by Definition 7}\ \square
\end{aligned}$$

We are now ready to define the generalized reduction scheme:

Definition 9 (Reduction)

$$\forall c\in\mathcal{GC}(\tau):\ \mathrm{red}^\tau(\overline{f})\circ c \quad=\quad f_c\circ\mathcal{E}^\tau_c(\mathrm{red}^\tau(\overline{f}))$$

For example, the following is the definition of the binary reduction:

$$\mathrm{red}^{T_1\times T_2}(\overline{f})\circ(C_1\times C_2) \quad=\quad f_{c_1\times c_2}\circ\mathcal{E}^{T_1\times T_2}_{c_1\times c_2}(\mathrm{red}^{T_1\times T_2}(\overline{f}))$$

where C_1 and C_2 are value constructors of T_1 and T_2, respectively.

For example, the binary reduction operator $F = \mathrm{red}^{list\times list}(f_{nn}, f_{nc}, f_{cn}, f_{cc})$ is defined as:

$$\begin{aligned}
F(\mathrm{Nil},\mathrm{Nil}) &= f_{nn}((),())\\
F(\mathrm{Nil},\mathrm{Cons}(b,s)) &= f_{nc}((),(b,s))\\
F(\mathrm{Cons}(a,l),\mathrm{Nil}) &= f_{cn}((a,l),())\\
F(\mathrm{Cons}(a,l),\mathrm{Cons}(b,s)) &= f_{cc}(a,(b,F(l,s)))
\end{aligned}$$

The following are examples of binary reductions:

$$\begin{aligned}
\mathrm{nateq}(x,y) &= \mathrm{red}^{nat\times nat}(\lambda((),()).\mathrm{True},\lambda((),j).\mathrm{False},\lambda(i,()).\mathrm{False},\lambda(r).r)\,(x,y)\\
\mathrm{listeq}(x,y) &= \mathrm{red}^{list\times list}(\lambda((),()).\mathrm{True},\lambda((),(b,s)).\mathrm{False},\lambda((a,l),()).\mathrm{False},\lambda(a,(b,r)).r\wedge(a=b))\,(x,y)\\
\mathrm{zip}(x,y) &= \mathrm{red}^{list\times list}(\lambda((),()).\mathrm{Nil},\lambda((),(b,s)).\mathrm{Nil},\lambda((a,l),()).\mathrm{Nil},\lambda(a,(b,r)).\mathrm{Cons}((a,b),r))\,(x,y)\\
\mathrm{firstn}(n,x) &= \mathrm{red}^{nat\times list}(\lambda((),()).\mathrm{Nil},\lambda((),(a,l)).\mathrm{Nil},\lambda(i,()).\mathrm{Nil},\lambda(a,r).\mathrm{Cons}(a,r))\,(n,x)\\
\mathrm{nth}(d)(x,n) &= \mathrm{red}^{list\times nat}(\lambda((),()).d,\lambda((),i).d,\lambda((a,l),()).a,\lambda(a,r).r)\,(x,n)\\
x\dot{-}y &= \mathrm{red}^{nat\times nat}(\lambda((),()).\mathrm{Zero},\lambda((),j).\mathrm{Zero},\lambda(i,()).\mathrm{Succ}(i),\lambda(r).r)\,(x,y)
\end{aligned}$$

3 Promotion Theorems

The general law which applies to all reductions is called the *general promotion theorem*. In this section we will present two special cases of the general promotion theorem, which are also the most common cases. The promotion theorem is presented and proved in its general form in the appendix. These are the unary and the binary promotion theorems (a unary function composed with one generalized reduction and a binary function composed with two generalized reductions).

The first promotion theorem is for the case of composing a unary function g with any n-ary reduction.

Theorem 1 (Unary Promotion Theorem)

$$\frac{\forall c \in \mathcal{GC}(\tau) : \ \phi_c \circ \mathcal{D}^\tau_c(g) \ = \ g \circ f_c}{g \circ \mathrm{red}^\tau(\overline{f}) \ = \ \mathrm{red}^\tau(\overline{\phi})}$$

Proof: Let $\eta = g \circ \mathrm{red}^\tau(\overline{f})$ and $c \in \mathcal{GC}(\tau)$. Then

$$\begin{array}{rcll} \eta \circ c & = & g \circ \mathrm{red}^\tau(\overline{f}) \circ c & \\ & = & g \circ f_c \circ \mathcal{E}^\tau_c(\mathrm{red}^\tau(\overline{f})) & \text{by Definition 9} \\ & = & \phi_c \circ \mathcal{D}^\tau_c(g) \circ \mathcal{E}^\tau_c(\mathrm{red}^\tau(\overline{f})) & \text{by premise} \\ & = & \phi_c \circ \mathcal{E}^\tau_c(g \circ \mathrm{red}^\tau(\overline{f})) & \text{by Property 3} \\ & = & \phi_c \circ \mathcal{E}^\tau_c(\eta) & \end{array}$$

Thus, by Definition 9, η is equal to $\mathrm{red}^\tau(\overline{\phi})$. □

If τ is the simple inductive type T, then the unary promotion theorem is identical to the simple promotion theorem for simple reductions, as it is described in [15].

For example, the unary promotion theorem for the simple type $T = \mathrm{bush}(\alpha)$ is:

$$\frac{\begin{array}{rcl} \phi_l(a) & = & g(f_l(a)) \\ \phi_b(\mathrm{map}^{list}(g)\, s) & = & g(f_b(s)) \end{array}}{g(\mathrm{red}^{bush}(f_l, f_b)\, x) \ = \ \mathrm{red}^{bush}(\phi_l, \phi_b)\, x}$$

And the unary promotion theorem for the generalized type $\tau = \mathrm{list}(\alpha) \times \mathrm{list}(\beta)$ is:

$$\frac{\begin{array}{rcl} \phi_{nn}((),()) & = & g(f_{nn}((),())) \\ \phi_{nc}((),(b,s)) & = & g(f_{nc}((),(b,s))) \\ \phi_{cn}((a,l),()) & = & g(f_{cn}((a,l),())) \\ \phi_{cc}(a,(b,g(r))) & = & g(f_{cc}(a,(b,r))) \end{array}}{g(\mathrm{red}^{list\times list}(f_{nn}, f_{nc}, f_{cn}, f_{cc})\,(x,y)) \ = \ \mathrm{red}^{list\times list}(\phi_{nn}, \phi_{nc}, \phi_{cn}, \phi_{cc})\,(x,y)}$$

The second promotion theorem is for the case of composing a binary function g with any two n-ary reductions.

Theorem 2 (Binary Promotion Theorem)

$$\frac{\forall c_1 \in \mathcal{GC}(\tau_1), \forall c_2 \in \mathcal{GC}(\tau_2) : \quad \phi_{c_1 \times c_2} \circ \mathcal{M}^{\tau_1 \times \tau_2}_{c_1 \times c_2}(g) \ = \ g \circ (f_{c_1} \times h_{c_2})}{g \circ (\mathrm{red}^{\tau_1}(\overline{f}) \times \mathrm{red}^{\tau_2}(\overline{h})) \ = \ \mathrm{red}^{\tau_1 \times \tau_2}(\overline{\phi})}$$

Proof: Let $F = \mathrm{red}^{\tau_1}(\overline{f})$, $H = \mathrm{red}^{\tau_2}(\overline{h})$, $\eta = g \circ (F \times H)$, and c_1 and c_2 are generalized constructors in $\mathcal{GC}(\tau_1)$ and $\mathcal{GC}(\tau_2)$. Then

$$\begin{array}{rcll} \eta \circ (c_1 \times c_2) & = & g \circ (F \times H) \circ (c_1 \times c_2) & \\ & = & g \circ ((F \circ c_1) \times (H \circ c_2)) & \\ & = & g \circ ((f_{c_1} \circ \mathcal{E}^{\tau_1}_{c_1}(F)) \times (h_{c_2} \circ \mathcal{E}^{\tau_2}_{c_2}(H))) & \text{by Definition 9} \\ & = & g \circ (f_{c_1} \times h_{c_2}) \circ (\mathcal{E}^{\tau_1}_{c_1}(F) \times \mathcal{E}^{\tau_2}_{c_2}(H)) & \\ & = & \phi_{c_1 \times c_2} \circ \mathcal{M}^{\tau_1 \times \tau_2}_{c_1 \times c_2}(g) \circ (\mathcal{E}^{\tau_1}_{c_1}(F) \times \mathcal{E}^{\tau_2}_{c_2}(H)) & \text{by premise} \\ & = & \phi_{c_1 \times c_2} \circ \mathcal{E}^{\tau_1 \times \tau_2}_{c_1 \times c_2}(g \circ (F \times H)) & \text{by Property 4} \\ & = & \phi_{c_1 \times c_2} \circ \mathcal{E}^{\tau_1 \times \tau_2}_{c_1 \times c_2}(\eta) & \end{array}$$

$$
\begin{array}{lcll}
\mathcal{N}[\![\mathcal{INV}(g)]\!] & \rightarrow & \textbf{raise}\ \text{inverse} & \textit{illegal use of inverse} \\
\mathcal{N}[\![v]\!] & \rightarrow & v & \textit{variable} \\
\mathcal{N}[\![c]\!] & \rightarrow & c & \textit{construction} \\
\mathcal{N}[\![(t_1,t_2)]\!] & \rightarrow & (\mathcal{N}[\![t_1]\!],\mathcal{N}[\![t_2]\!]) & \textit{pair} \\
\mathcal{N}[\![\lambda x.e]\!] & \rightarrow & \lambda x.\mathcal{N}[\![e]\!] & \textit{abstraction} \\
\mathcal{N}[\![\mathrm{red}^\tau(f_1,\ldots,f_n)]\!] & \rightarrow & \mathrm{red}^\tau(\mathcal{N}[\![f_1]\!],\ldots,\mathcal{N}[\![f_n]\!]) & \textit{reduction} \\
\mathcal{N}[\![\mathrm{red}^\tau(\overline{f})\,(c\,t)]\!] & \rightarrow & \mathcal{N}[\![f_c(\mathcal{E}^\tau_c(\mathrm{red}^\tau(\overline{f}))\,t)]\!] & \textit{combinator reduction} \\
\mathcal{N}[\![g(\mathrm{red}^\tau(\overline{f})\,t)]\!] & \rightarrow & \left\{\begin{array}{l} \mathrm{red}^\tau(\overline{\phi})\,(\mathcal{N}[\![t]\!]) \\ \quad \textbf{where}\ \forall c\in\mathcal{GC}(\tau): \\ \quad \phi_c=\lambda\overline{x}.\mathcal{N}[\![g(f_c(\mathcal{D}^\tau_c(\mathcal{INV}(g))\,\overline{x}))]\!] \\ \textbf{handle}\ \text{inverse}\Rightarrow\mathcal{N}[\![g]\!](\mathcal{N}[\![\mathrm{red}^\tau(\overline{f})\,t]\!]) \end{array}\right. & \textit{unary promotion} \\
\mathcal{N}[\![g(\mathrm{red}^{\tau_1}(\overline{f})\,t_1,\mathrm{red}^{\tau_2}(\overline{h})\,t_2)]\!] & \rightarrow & \left\{\begin{array}{l} \mathrm{red}^{\tau_1\times\tau_2}(\overline{\phi})(\mathcal{N}[\![t_1]\!],\mathcal{N}[\![t_2]\!]) \\ \quad \textbf{where}\ \forall c_1\in\mathcal{GC}(\tau_1),\ \forall c_2\in\mathcal{GC}(\tau_2): \quad \textit{binary promotion} \\ \quad \phi_{c_1\times c_2}=\lambda(\mathcal{E}^{\tau_1\times\tau_2}_{c_1\times c_2}(\#)(\overline{x_1},\overline{x_2})). \\ \qquad\qquad \mathcal{N}[\![g(f_{c_1}(\mathcal{E}^{\tau_1}_{c_1}(\mathcal{INV}(g))\,\overline{x_1}),h_{c_2}(\mathcal{E}^{\tau_2}_{c_2}(\mathcal{INV}(g))\,\overline{x_2}))]\!] \\ \textbf{handle}\ \text{inverse}\Rightarrow\mathcal{N}[\![g]\!](\mathcal{N}[\![\mathrm{red}^{\tau_1}(\overline{f})\,t_1]\!],\mathcal{N}[\![\mathrm{red}^{\tau_2}(\overline{h})\,t_2]\!]) \end{array}\right. & \\
\mathcal{N}[\![g(\mathcal{INV}(g)\,x)]\!] & \rightarrow & x & \textit{unary elimination} \\
\mathcal{N}[\![g(\mathcal{INV}(g)\,x_1,\mathcal{INV}(g)\,x_2)]\!] & \rightarrow & \#(x_1,x_2)\ \equiv\ x_1\_x_2 & \textit{binary elimination} \\
\mathcal{N}[\![(\lambda v.e)\,t]\!] & \rightarrow & \mathcal{N}[\![\mathrm{beta}(v,e,t)]\!] & \textit{β reduction} \\
\mathcal{N}[\![f\,e]\!] & \rightarrow & \mathcal{N}[\![f]\!](\mathcal{N}[\![e]\!]) & \textit{application}
\end{array}
$$

Figure 1: The Normalization Algorithm

Thus, by Definition 9, η is equal to $\mathrm{red}^{\tau_1\times\tau_2}(\overline{\phi})$. □

For example, for $T_1 = \mathrm{list}(\alpha)$ and $T_2 = \mathrm{nat}$ the binary promotion theorem is:

$$
\frac{\begin{array}{lcl} \phi_{nz}((),()) & = & g(f_n(),h_z()) \\ \phi_{ns}((),s) & = & g(f_n(),h_s(s)) \\ \phi_{cz}((a,r),()) & = & g(f_c(a,r),h_z()) \\ \phi_{cs}(a,g(r,s)) & = & g(f_c(a,r),h_s(s)) \end{array}}{g(\mathrm{red}^{list}(f_n,f_c)\ x,\mathrm{red}^{nat}(h_z,h_s)\ y)\ =\ \mathrm{red}^{list\times nat}(\phi_{nz},\phi_{ns},\phi_{cz},\phi_{cs})\ (x,y)}
$$

4 The Normalization Algorithm

In this section we present our program optimization algorithm. It uses the unary and binary promotion theorems effectively to perform loop fusion. This algorithm, called the *normalization algorithm*, is presented in Figure 1. Term $\mathcal{INV}(g)$ denotes a special intermediate term that should not appear in the normalized term. To enforce this property, the first rule raises an exception if the normalization algorithm encounters such a term. Normally, $\mathcal{INV}(g)$ is cancelled by g in the *elimination phases* and no exception is raised. If an exception is raised, then it is caught by the undergoing *promotion phases* and no loop fusion is performed. Otherwise, the promotion theorem is used to fuse the two nested reductions into one. Note that the *binary promotion phase* constructs the lambda variables of the new accumulating function by using variable name concatenation $\#(x_1,x_2) = x_1\_x_2$. Variables $\overline{x}$ in the unary promotion and $\overline{x_1}$ and $\overline{x_2}$ in the binary promotion phase are new variable names.

For example, the following is an instance of the unary promotion phase of the normalization algorithm:

$$
\mathcal{N}[\![g(\mathrm{red}^{list}(f_n,f_c)\,x)]\!]\ \rightarrow\ \mathrm{red}^{list}(\phi_n,\phi_c)\,(\mathcal{N}[\![x]\!])
$$

where

$$\begin{array}{lcl} \phi_n & = & \lambda().\mathcal{N}[\![g(f_n())]\!] \\ \phi_c & = & \lambda(a,s).\mathcal{N}[\![g(f_c(a,\mathcal{IN\!V}(g)\,s))]\!] \end{array}$$

The following is an instance of the binary promotion phase of the normalization algorithm:

$$\mathcal{N}[\![g(\mathrm{red}^{list}(f_n,f_c)\;x,\mathrm{red}^{nat}(h_z,h_s)\;y)]\!] \;\;\rightarrow\;\; \mathrm{red}^{list\times nat}(\phi_{nz},\phi_{ns},\phi_{cz},\phi_{cs})\;(\mathcal{N}[\![x]\!],\mathcal{N}[\![y]\!])$$

where

$$\begin{array}{lcl} \phi_{nz} & = & \lambda((),()).\mathcal{N}[\![g(f_n(),h_z())]\!] \\ \phi_{ns} & = & \lambda((),s).\mathcal{N}[\![g(f_n(),h_s(s))]\!] \\ \phi_{cz} & = & \lambda((a,r),()).\mathcal{N}[\![g(f_c(a,r),h_z())]\!] \\ \phi_{cs} & = & \lambda(a,r\_s).\mathcal{N}[\![g(f_c(a,\mathcal{IN\!V}(g)\,r),h_s(\mathcal{IN\!V}(g)\,s))]\!] \end{array}$$

It is not easy to prove the correctness of the normalization algorithm. In general, we need to define formally the meaning function that maps terms into values and prove that the normalization algorithm always preserves meaning. The only mechanisms we have to prove this are the promotion theorems. We will not present the detailed proof here. Instead we will present a sketch of the proof. The detailed correctness proof for the normalization algorithm which includes only the simple promotion theorem can be found elsewhere [16].

Theorem 3 (Correctness of the Normalization Algorithm) *The normalization algorithm always preserves the meaning of a term.*

Proof sketch: All the transformation rules of the normalization algorithm can be easily proved to preserve the meaning of a term, except for the two promotion laws. We consider the unary promotion phase first. There are two cases for the computation of the new accumulating functions ϕ_c: if any of the computations $\mathcal{N}[\![g(f_c(\mathcal{D}_c^\tau(\mathcal{IN\!V}(g))\,\overline{x}))]\!]$ raises the *inverse* exception during the normalization process, then there will be no fusion performed. Otherwise, g is fused with $\mathcal{IN\!V}(g)$ during the normalization process. To see why $\mathcal{N}[\![g(f_c(\mathcal{D}_c^\tau(\mathcal{IN\!V}(g))\,\overline{x}))]\!]$ computes ϕ_c we use the unary promotion theorem:

$$\begin{array}{lll} \forall c\in\mathcal{GC}(\tau): & \phi_c\circ\mathcal{D}_c^\tau(g) \;=\; g\circ f_c & \\ \Leftrightarrow & \phi_c\circ\mathcal{D}_c^\tau(g)\circ\mathcal{E}_c^\tau(\mathcal{IN\!V}(g)) \;=\; g\circ f_c\circ\mathcal{E}_c^\tau(\mathcal{IN\!V}(g)) & \\ \Leftrightarrow & \phi_c\circ\mathcal{E}_c^\tau(g\circ\mathcal{IN\!V}(g)) \;=\; g\circ f_c\circ\mathcal{E}_c^\tau(\mathcal{IN\!V}(g)) & \textit{by Property (3)} \\ \Leftrightarrow & \phi_c \;=\; g\circ f_c\circ\mathcal{E}_c^\tau(\mathcal{IN\!V}(g)) & \textit{by unary elimination} \end{array}$$

where $g\circ\mathcal{IN\!V}(g)$ was cancelled out in the unary elimination phase. In order to prove that the binary promotion phase of the normalization algorithm is correct, we use the binary promotion theorem. Let $c_1\in\mathcal{GC}(\tau_1)$ and $c_2\in\mathcal{GC}(\tau_2)$. Then from the binary promotion theorem we have:

$$\begin{array}{ll} & \phi_{c_1\times c_2}\circ\mathcal{M}_{c_1\times c_2}^{\tau_1\times\tau_2}(g) \;=\; g\circ(f_{c_1}\times h_{c_2}) \\ \Leftrightarrow & \phi_{c_1\times c_2}\circ\mathcal{M}_{c_1\times c_2}^{\tau_1\times\tau_2}(g)\circ(\mathcal{E}_{c_1}^{\tau_1}(\mathcal{IN\!V}(g))\times\mathcal{E}_{c_2}^{\tau_2}(\mathcal{IN\!V}(g))) \;=\; g\circ(f_{c_1}\times h_{c_2})\circ(\mathcal{E}_{c_1}^{\tau_1}(\mathcal{IN\!V}(g))\times\mathcal{E}_{c_2}^{\tau_2}(\mathcal{IN\!V}(g))) \\ \Leftrightarrow & \phi_{c_1\times c_2}\circ\mathcal{E}_{c_1\times c_2}^{\tau_1\times\tau_2}(g\circ(\mathcal{IN\!V}(g)\times\mathcal{IN\!V}(g))) \;=\; g\circ(f_{c_1}\times h_{c_2})\circ(\mathcal{E}_{c_1}^{\tau_1}(\mathcal{IN\!V}(g))\times\mathcal{E}_{c_2}^{\tau_2}(\mathcal{IN\!V}(g))) \\ \Leftrightarrow & \phi_{c_1\times c_2}\circ\mathcal{E}_{c_1\times c_2}^{\tau_1\times\tau_2}(\#) \;=\; g\circ(f_{c_1}\times h_{c_2})\circ(\mathcal{E}_{c_1}^{\tau_1}(\mathcal{IN\!V}(g))\times\mathcal{E}_{c_2}^{\tau_2}(\mathcal{IN\!V}(g))) \end{array}$$

□

4.1 Example of a Normalization by Unary Promotion

We will improve length(zip(x,y)), where:

$$\mathrm{length}(x) \;=\; \mathrm{red}^{list}(\lambda().\mathrm{Zero},\lambda(a,r).\mathrm{Succ}(r))\,x$$

$$\mathrm{zip}(x,y) \;=\; \mathrm{red}^{list\times list}(f_{nn},f_{nc},f_{cn},f_{cc})\,(x,y) \quad \text{where} \quad \left\{\begin{array}{lll} f_{nn} \;=\; \lambda((),()).\mathrm{Nil} & (u1) \\ f_{nc} \;=\; \lambda((),(b,s)).\mathrm{Nil} & (u2) \\ f_{cn} \;=\; \lambda((a,l),()).\mathrm{Nil} & (u3) \\ f_{cc} \;=\; \lambda(a,(b,r)).\mathrm{Cons}((a,b),r) & (u4) \end{array}\right.$$

From the unary promotion phase of the normalization algorithm:

$$\mathcal{N}[\![\mathrm{length}(\mathrm{red}^{list\times list}(f_{nn},f_{nc},f_{cn},f_{cc})\,(x,y))]\!] \;\rightarrow\; \mathrm{red}^{list\times list}(\phi_{nn},\phi_{nc},\phi_{cn},\phi_{cc})\,(x,y)$$

where:

$$1)\quad \phi_{nn} = \lambda().\text{length}(f_{nn}()) = \lambda().\text{length}(\text{Nil}) = \lambda().\text{Zero}$$

$$2)\quad \phi_{nc} = \lambda((),(b,s)).\text{length}(f_{nc}((),(b,s))) = \lambda((),(b,s)).\text{length}(\text{Nil}) = \lambda((),(b,s)).\text{Zero}$$

$$3)\quad \phi_{cn} = \lambda((a,l),()).\text{length}(f_{nn}((a,l),())) = \lambda((a,l),()).\text{length}(\text{Nil}) = \lambda((a,l),()).\text{Zero}$$

$$\begin{aligned} 4)\quad \phi_{cc} &= \lambda(a,(b,r)).\text{length}(f_{cc}(a,(b,\mathcal{INV}(\text{length})(r)))) \\ &= \lambda(a,(b,r)).\text{length}(\text{Cons}((a,b),\mathcal{INV}(\text{length})(r))) && \textit{(by (u4))} \\ &= \lambda(a,(b,r)).\text{Succ}(\text{length}(\mathcal{INV}(\text{length})(r))) && \textit{(by combinator reduction)} \\ &= \lambda(a,(b,r)).\text{Succ}(r) && \textit{(by unary elimination)} \end{aligned}$$

Therefore:

$$\mathcal{N}[\![\text{length}(\text{zip}(x,y))]\!] \rightarrow \text{red}^{list\times list}(\lambda().\text{Zero},\lambda((),(b,s)).\text{Zero},\lambda((a,l),()).\text{Zero},\lambda(a,(b,r)).\text{Succ}(r))(x,y)$$

4.2 Example of a Normalization by Binary Promotion

We will normalize $\text{zip}(\text{map}(f)(x),\text{map}(g)(y))$, which is a binary reduction applied to two unary reductions, where:

$$\text{map}(f)(x) = \text{red}^{list}(\lambda().\text{Nil},\lambda(a,r).\text{Cons}(f\,a,r))\,x$$

The binary promotion part of the normalization algorithm applied to this case is:

$$\begin{aligned} &\mathcal{N}[\![\text{zip}(\text{red}^{list}(\lambda().\text{Nil},\lambda(a,r).\text{Cons}(f\,a,r))\,x,\text{red}^{list}(\lambda().\text{Nil},\lambda(b,s).\text{Cons}(g\,b,s))\,y)]\!] \\ &\qquad\rightarrow \text{red}^{list\times list}(\phi_{nn},\phi_{nc},\phi_{cn},\phi_{cc})\,(x,y) \end{aligned}$$

From the binary promotion phase of the normalization algorithm we have:

$$1)\quad \phi_{nn} = \lambda((),()).\text{zip}(\text{Nil},\text{Nil}) = \lambda((),()).\text{Nil}$$

$$2)\quad \phi_{nc} = \lambda((),(b,s)).\text{zip}(\text{Nil},\text{Cons}(g\,b,s)) = \lambda((),(b,s)).\text{Nil}$$

$$3)\quad \phi_{cn} = \lambda((a,r),()).\text{zip}(\text{Cons}(f\,a,r),\text{Nil}) = \lambda((a,r),()).\text{Nil}$$

$$\begin{aligned} 4)\quad \phi_{cc} &= \lambda(a,(b,r\_s)).\text{zip}(\text{Cons}(f\,a,\mathcal{INV}(\text{zip})\,r),\text{Cons}(g\,b,\mathcal{INV}(\text{zip})\,s)) \\ &= \lambda(a,(b,r\_s)).\text{Cons}((f\,a,g\,b),\text{zip}(\mathcal{INV}(\text{zip})\,r,\mathcal{INV}(\text{zip})\,s)) \\ &= \lambda(a,(b,r\_s)).\text{Cons}((f\,a,g\,b),r\_s) \end{aligned}$$

Therefore, $\text{zip}(\text{map}(f)(x),\text{map}(g)(y))$ is normalized to

$$\text{red}^{list\times list}(\lambda((),()).\text{Nil},\lambda((),(b,s)).\text{Nil},\lambda((a,r),()).\text{Nil},\lambda(a,(b,r\_s)).\text{Cons}((f\,a,g\,b),r\_s))(x,y)$$

5 Related Work

To our knowledge no other work deals with generic recursion schemes over multiple structures. For simple inductions our work is closely related to Wadler's work on listlessness and deforestation [18, 8, 17] and to Chin's work on fusion [2]. Deforestation works on all first order treeless terms. A treeless term is one which is exactly analogous to a safe term, but is described in a much different manner due to the lack of structure imposed on such terms. Chin generalizes Wadler's techniques to all first order programs, not just treeless ones, by recognizing and skipping over terms to which his techniques do not apply. His work also applies to higher order programs in general. This is accomplished by a higher order removal phase, which first removes some higher order functions from a program. Those not removed are recognizable and are simply skipped over in the improvement phase. The application domain of our fusion algorithm is more restricted than the domain of all these methods, but our algorithm is more effective since it is fully automated. In [9] a new, simple, but very effective, automatic technique is presented for implementing deforestation in a compiler. This method requires that each list-producing functions is expressed as a *build* call and each list-consuming functions is expressed as a *foldr* call. The foldr operator is similar to our list reduction while

the build operator is a dual-like function of foldr (this technique is an automation of the *HyloSplit* theorem of Meijer et al. [14]). The technique simply fuses adjacent foldr-build pairs by eliminating them completely ($(\text{foldr}\, f) \circ (\text{build}\, g) = g \circ f$). We believe that this method could be more effective if folds are promoted downwards or builds are promoted upwards in a term until they fuse. This can be achieved effectively by applying promotion theorems.

Our stereotyped recursion schemes as well as the promotion theorems are highly influenced by the Squiggol school of program construction [13, 14, 12, 1]. Their goal is to construct a calculus of programs based on some well-behaved recursion schemes, in which their inductive laws, proved once and for all in their generic form, can be instantiated and used for calculating program transformation as well as for proving properties about programs without the need for discovering new laws or using explicit induction. The promotion theorems are examples of a large class of theorems that come for free. We believe that our notation, which is based on calculus of construction, is more intuitive to functional programmers than their formalism (also called the Bird-Meertens Formalism), which is based on category theory. Even though their work is more general than ours, we provide a fully automated system that use the promotion laws effectively.

6 Conclusion

The functional programming style has been criticized because of its waste of resources caused by the building of intermediate data structures, unused closures, and garbage collection. We believe that this is not caused by the functional style per se, but by the use of unrestricted recursion which makes it difficult to validate the application of well known optimizations in unstructured programs.

Recent language proposals to program with the explicit structure of generic recursion schemes are an attempt to circumnavigate these problems. This paper provides a first step towards automatic optimization and compilation for such languages. To our knowledge the algorithm presented here is the first algorithm (not based upon a fixed set of patterns and datatypes) that automatically performs fusion without a memoization phase, and which deals with multiple inductions.

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A The General Promotion Theorem

In this appendix we generalize the promotion theorem to capture any combination of n-ary reductions. In order to do that, we generalize the combinators $\mathcal{E}^{\tau}_{c}$ and $\mathcal{M}^{\tau}_{c}$ as $\mathcal{E}^{\tau/\tau'}_{c}$ and $\mathcal{M}^{\tau/\tau'}_{c}$:

$$\begin{array}{lcl}
\mathcal{E}^{\tau/T}_{c}(f) & = & \mathcal{E}^{\tau}_{c}(f) \\
\mathcal{E}^{\tau_1\times\tau_2/\tau_1'\times\tau_2'}_{c_1\times c_2}(f_1\times f_2) & = & \mathcal{E}^{\tau_1/\tau_1'}_{c_1}(f_1)\times\mathcal{E}^{\tau_2/\tau_2'}_{c_2}(f_2) \\
 & & \\
\mathcal{M}^{\tau/T}_{c}(f) & = & \mathcal{D}^{\tau}_{c}(f) \\
\mathcal{M}^{\tau_1\times\tau_2/\tau_1'\times\tau_2'}_{c_1\times c_2}(f) & = & \lambda(x,y).\mathcal{M}^{\tau_1/\tau_1'}_{c_1}(\lambda z.\mathcal{M}^{\tau_2/\tau_2'}_{c_2}(\lambda w.f(z,w))\,y)\,x
\end{array}$$

Definition 10 (General Reduction) *A function f of type $\tau \to \tau'$ is a general reduction (denoted as $\mathcal{G}^{\tau\to\tau'}$) if it is derived from the following rules:*

$$\begin{array}{ll}
\mathrm{id}^{T} \in \mathcal{G}^{T\to T} & \text{where } \mathrm{id}^{T} \text{ is the identity for type } T \\
\mathrm{red}^{\tau}(\overline{f}) \in \mathcal{G}^{\tau\to T} & \text{if } \mathrm{red}^{\tau}(\overline{f}) \text{ is of type } \tau \to T \\
f_1\times f_2 \in \mathcal{G}^{(\tau_1\times\tau_2)\to(\tau_1'\times\tau_2')} & \text{if } f_1 \in \mathcal{G}^{\tau_1\to\tau_1'} \text{ and } f_2 \in \mathcal{G}^{\tau_2\to\tau_2'}
\end{array}$$

For example, $\mathrm{length}\times\mathrm{id}^{nat}$ is a general reduction from $\mathcal{G}^{(\mathrm{list}\times\mathrm{nat})\to(\mathrm{nat}\times\mathrm{nat})}$. We will present a promotion theorem for any composition $g\circ f$, where $g:\tau'\to\alpha$ and $f\in\mathcal{G}^{\tau\to\tau'}$.

Lemma 1 *For any $f\in\mathcal{G}^{\tau\to\tau'}$, $g:\tau'\to\alpha$, and $c\in\mathcal{GC}(\tau)$:*

$$\mathcal{E}^{\tau}_{c}(g\circ f) \quad = \quad \mathcal{M}^{\tau/\tau'}_{c}(g)\circ\mathcal{E}^{\tau/\tau'}_{c}(f)$$

Proof: If $\tau' = T$ and $f = \mathrm{id}$ or $f = \mathrm{red}^\tau(\overline{f})$ we have $\mathcal{E}^\tau_c(g \circ f) = \mathcal{D}^\tau_c(g) \circ \mathcal{E}^\tau_c(f)$, which is true because of Property 3. We assume the theorem is true for τ_1 and τ_2. For $(f_1 \times f_2) \in \mathcal{G}^{(\tau_1 \times \tau_2) \to (\tau_1' \times \tau_2')}$ and $c = c_1 \times c_2$ we have:

$$\begin{aligned}
\mathcal{E}^{\tau_1 \times \tau_2}_{c_1 \times c_2}(g \circ (f_1 \times f_2)) &= \lambda(x,y).\mathcal{E}^{\tau_1}_{c_1}(\lambda z.\mathcal{E}^{\tau_2}_{c_2}(\lambda w.g(f_1(z), f_2(w)))\, y)\, x \\
&= \lambda(x,y).\mathcal{M}^{\tau_1/\tau_1'}_{c_1}(\lambda z.\mathcal{M}^{\tau_2/\tau_2'}_{c_2}(\lambda w.g(z,w))\,(\mathcal{E}^{\tau_2/\tau_2'}_{c_2}(f_2)\, y))\,(\mathcal{E}^{\tau_1/\tau_1'}_{c_1}(f_1)\, x) \\
&= (\lambda(x,y).\mathcal{M}^{\tau_1/\tau_1'}_{c_1}(\lambda z.\mathcal{M}^{\tau_2/\tau_2'}_{c_2}(\lambda w.g(z,w))\, y)\, x) \circ (\mathcal{E}^{\tau_1/\tau_1'}_{c_1}(f_1) \times \mathcal{E}^{\tau_2/\tau_2'}_{c_2}(f_2)) \\
&= \mathcal{M}^{\tau_1 \times \tau_2/\tau_1' \times \tau_2'}_{c_1 \times c_2}(g) \circ \mathcal{E}^{\tau_1 \times \tau_2/\tau_1' \times \tau_2'}_{c_1 \times c_2}(f_1 \times f_2) \qquad \square
\end{aligned}$$

The combinator $\mathcal{I}_c$, for $f \in \mathcal{G}^{\tau \to \tau'}$ and $c \in \mathcal{GC}(\tau)$, is defined as follows:

$$\begin{aligned}
\mathcal{I}_c(\mathrm{id}^T) &= c \\
\mathcal{I}_c(\mathrm{red}^\tau(\overline{f})) &= f_c \\
\mathcal{I}_{c_1 \times c_2}(f_1 \times f_2) &= \mathcal{I}_{c_1}(f_1) \times \mathcal{I}_{c_2}(f_2)
\end{aligned}$$

Note that the first equation may be derived from the second, since for any inductive type T we have $\mathrm{id} = \mathrm{red}^T(\overline{f})$, where $f_c = c$ for any value constructor c of T.

Lemma 2 *For any $f \in \mathcal{G}^{\tau \to \tau'}$ and $c \in \mathcal{GC}(\tau)$:*

$$f \circ c = \mathcal{I}_c(f) \circ \mathcal{E}^{\tau/\tau'}_c(f)$$

Proof: If $\tau = \tau' = T$ and $f = \mathrm{id}$ then $\mathcal{I}_c(f) \circ \mathcal{E}^{\tau/\tau'}_c(f) = c \circ E^T_c(f) = c = f \circ c$. For $\tau' = T$ and $f = \mathrm{red}^\tau(\overline{f})$ we have $\mathcal{I}_c(f) = f_c$. Then $f \circ c = f_c \circ \mathcal{E}^\tau_c(f)$, which is the definition of reduction. We assume the theorem is true for τ_1 and τ_2. For $f = (f_1 \times f_2) \in \mathcal{G}^{(\tau_1 \times \tau_2) \to (\tau_1' \times \tau_2')}$ and $c = c_1 \times c_2$ we have:

$$\begin{aligned}
f \circ c &= (f_1 \times f_2) \circ (c_1 \times c_2) \\
&= (f_1 \circ c_1) \times (f_2 \circ c_2) \\
&= (\mathcal{I}_{c_1}(f_1) \circ \mathcal{E}^{\tau_1/\tau_1'}_{c_1}(f_1)) \times (\mathcal{I}_{c_2}(f_2) \circ \mathcal{E}^{\tau_2/\tau_2'}_{c_2}(f_2)) \\
&= (\mathcal{I}_{c_1}(f_1) \times \mathcal{I}_{c_2}(f_2)) \circ (\mathcal{E}^{\tau_1/\tau_1'}_{c_1}(f_1) \times \mathcal{E}^{\tau_2/\tau_2'}_{c_2}(f_2)) \\
&= \mathcal{I}_c(f) \circ \mathcal{E}^{\tau/\tau'}_c(f) \qquad \square
\end{aligned}$$

Theorem 4 (General Promotion Theorem) *Let $f \in \mathcal{G}^{\tau \to \tau'}$ and $g : \tau' \to \alpha$. Then*

$$\frac{\forall c \in \mathcal{GC}(\tau) :\ \phi_c \circ \mathcal{M}^{\tau/\tau'}_c(g) \ =\ g \circ \mathcal{I}_c(f)}{g \circ f \ =\ \mathrm{red}^\tau(\overline{\phi})}$$

Proof: We will use Lemmas 1 and 2. Let $\eta = g \circ f$ and $c \in \mathcal{GC}(\tau)$. Then

$$\begin{aligned}
\eta \circ c &= g \circ f \circ c \\
&= g \circ \mathcal{I}_c(f) \circ \mathcal{E}^{\tau/\tau'}_c(f) && \text{by Lemma 2} \\
&= \phi_c \circ \mathcal{M}^{\tau/\tau'}_c(g) \circ \mathcal{E}^{\tau/\tau'}_c(f) && \text{by premise} \\
&= \phi_c \circ \mathcal{E}^\tau_c(g \circ f) && \text{by Lemma 1} \\
&= \phi_c \circ \mathcal{E}^\tau_c(\eta)
\end{aligned}$$

Thus, η is the reduction $\mathrm{red}^\tau(\overline{\phi})$. $\square$

The following two rules extend the normalization algorithm, presented in Figure 1, with a general promotion phase. Therefore, the unary and the binary promotion phases as well as the unary and binary elimination phases of this algorithm can be replaced by the following rules:

$$\mathcal{N}[\![g(f(\overline{t}))]\!] \ \to\ \begin{cases} \mathrm{red}^\tau(\overline{\phi})\,(\mathcal{N}[\![\overline{t}]\!]) \\ \quad \textbf{where } f \in \mathcal{G}^{\tau \to \tau'},\ g : \tau' \to \alpha, \text{ and } \forall c \in \mathcal{GC}(\tau) : \\ \quad \phi_c = \lambda(\mathcal{E}^\tau_c(\#)\,\overline{x}).\mathcal{N}[\![g(\mathcal{I}_c(f)(\mathcal{EI}^\tau_c(g)\,\overline{x}))]\!] \\ \textbf{handle } \text{inverse} \Rightarrow \mathcal{N}[\![g]\!](\mathcal{N}[\![f(t)]\!]) \end{cases} \qquad \textit{general promotion}$$

$$\mathcal{N}[\![g(\mathcal{EI}^\tau_c(g)\,\overline{x})]\!] \ \to\ \#(\overline{x}) \qquad \textit{general elimination}$$

where $\mathcal{EI}^{\tau}_{c}(g)$ creates multiple copies of the function $\mathcal{INV}(g)$ into a product that has the same shape as c:

$$\begin{array}{lcl} \mathcal{EI}^{\tau}_{c}(g) & = & \mathcal{E}^{\tau}_{c}(\mathcal{INV}(g)) \\ \mathcal{EI}^{\tau_1\times\tau_2}_{c_1\times c_2}(g) & = & \mathcal{EI}^{\tau_1}_{c_1\times c_2}(g) \times \mathcal{EI}^{\tau_2}_{c_1\times c_2}(g) \end{array}$$

For example, we will normalize $\mathrm{nth}(d)(\mathrm{append}(x,y),n)$, where:

$$\mathrm{nth}(d)(x,n) \quad = \quad \mathrm{red}^{list\times nat}(\lambda((),()).d, \lambda((),i).d, \lambda((a,l),()).a, \lambda(a,r).r)\,(x,n)$$

We have $f \in \mathcal{G}^{(list\times nat)\mapsto(list\times nat)}$ and $f = \mathrm{red}^{list}(\lambda().y, \lambda(a,r).\mathrm{Cons}(a,r)) \times \mathrm{id}^{nat}$.

The general promotion part of the normalization algorithm applied to this case is:

$$\mathcal{N}[\![\mathrm{nth}(d)(\mathrm{red}^{list}(\lambda().y, \lambda(a,r).\mathrm{Cons}(a,r))\,x, \mathrm{id}^{nat}(n))]\!] \quad \rightarrow \quad \mathrm{red}^{list\times nat}(\phi_{nz}, \phi_{ns}, \phi_{cz}, \phi_{cs})\,(x,n)$$

where

1) $\phi_{nz} = \lambda((),()).\mathrm{nth}(d)(y, \mathrm{Zero})$

2) $\phi_{ns} = \lambda((),i).\mathrm{nth}(d)(y, \mathrm{Succ}(i))$

3) $\phi_{cz} = \lambda((a,r),()).\mathrm{nth}(d)(\mathrm{Cons}(a,r), \mathrm{Zero}) \;=\; \lambda((a,r),()).a$

4) $\begin{array}{lcl} \phi_{cs} & = & \lambda(a,r\_i).\mathrm{nth}(d)(\mathrm{Cons}(a, \mathcal{INV}(\mathrm{nth})\,r), \mathrm{Succ}(\mathcal{INV}(\mathrm{nth})\,i)) \\ & = & \lambda(a,r\_i).\mathrm{nth}(d)(\mathcal{INV}(\mathrm{nth})\,r, \mathcal{INV}(\mathrm{nth})\,i) \;=\; \lambda(a,r\_i).r\_i \end{array}$